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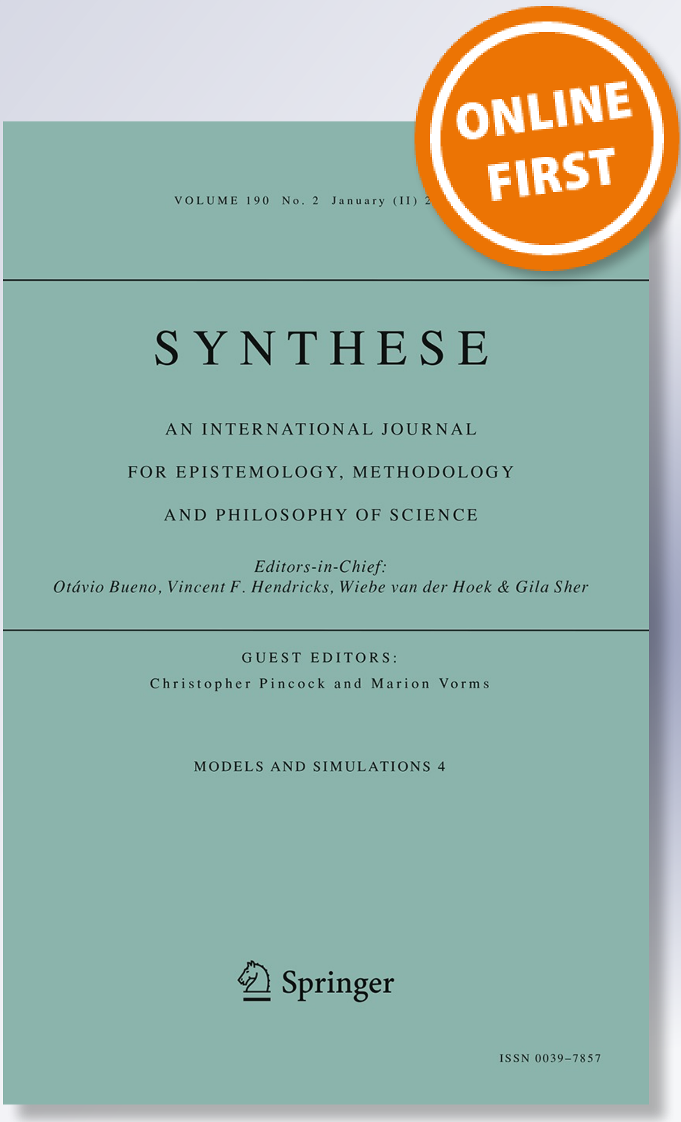

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Abstract In the literature, different axiomatizations of Public Announcement Logic (PAL) have been proposed. Most of these axiomatizations share a "core set" of the so-called "reduction axioms". In this paper, by designing non-standard Kripke semantics for the language of PAL, we show that the proof system based on this core set of axioms does not completely axiomatize PAL without additional axioms and rules. In fact, many of the intuitive axioms and rules we took for granted could not be derived from the core set. Moreover, we also propose and advocate an alternative yet meaningful axiomatization of PAL without the reduction axioms. The completeness is proved directly by a detour method using the canonical model where announcements are treated as merely labels for modalities as in normal modal logics. This new axiomatization and its completeness proof may sharpen our understanding of PAL and can be adapted to other dynamic epistemic logics.

Keywords Public announcement logic · Reduction axioms · Composition axiom · Dynamic epistemic logic · Completeness · Epistemic temporal logic

1 Introduction

The last two decades have witnessed the rapid developments of *Dynamic Epistemic Logic* (DEL) as a field which includes various modal logics based on the same central

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idea: actions as updaters of epistemic models (cf. van Ditmarsch et al. 2007; van Benthem 2011). Due to its flexibility in modelling various communicative actions, DEL has been applied to many related fields such as philosophy, artificial intelligence, computer science and linguistics. In recent years, dozens of new DEL-stlye logics were proposed and studied. In this paper, however, we would like to go back to the origin of DEL to examine and clarify some basic axiomatization results and techniques.

The initiative of DEL dates back to the late 1980s and 1990s when *Public Announcement Logic* (PAL) was independently proposed and studied by Plaza (1989), and Gerbrandy and Groeneveld (1997). PAL is a logic for reasoning about knowledge changes under public communications. Many techniques that have been used in the current developments of DEL are inherited from the very early works on PAL, e.g., the use of *reduction axioms* in axiomatizations of DEL-style logics. Let us briefly review the syntax and semantics of PAL first.

1.1 Public announcement logic

Given a non-empty set **P** of basic proposition letters, the language of *Public Announcement Logic* (PAL) (Plaza 1989; Gerbrandy and Groeneveld 1997) is usually presented as follows[1]:

$$\phi ::= \top \mid p \mid \neg\phi \mid \phi \wedge \phi \mid \Box\phi \mid [\phi]\phi$$

where $p \in \mathbf{P}$. As usual, we define $\bot, \phi \vee \psi, \phi \to \psi$ and $\langle\psi\rangle\phi$ as the abbreviations of $\neg\top$, $\neg(\neg\phi \wedge \neg\psi)$, $\neg\phi \vee \psi$ and $\neg[\psi]\neg\phi$ respectively. The original reading of $\Box\phi$ as in Plaza (1989) is that "I know ϕ" and $[\psi]\phi$ expresses "After announcing ψ publicly, ϕ holds." Following the epistemic tradition, in this paper we call the $[\phi]$-free fragment of the PAL language the language of *Epistemic Logic* (EL).

The language of PAL is interpreted on *Kripke models*. A Kripke model over a non-empty set **P** of basic propositions is a triple $(S, \to, V)$ where S is a non-empty set of possible worlds, $\to \subseteq S \times S$ is a binary relation over S and $V : \mathbf{P} \to 2^S$ is a valuation function assigning each basic proposition letter a set of worlds where it is true. Despite the epistemic setting in which PAL was initially proposed, in this paper, for technical generality, we *do not* restrict ourselves to epistemic (**S5**) models unless specified. Given a Kripke model $\mathcal{M} = (S, \to, V)$ over **P**, the truth value of PAL formulas at a state s in $\mathcal{M}$ is defined as follows:

$$\begin{array}{rcl}
\mathcal{M}, s \vDash \top & \Leftrightarrow & \text{always} \\
\mathcal{M}, s \vDash p & \Leftrightarrow & s \in V(p) \\
\mathcal{M}, s \vDash \neg\phi & \Leftrightarrow & \mathcal{M}, s \nvDash \phi \\
\mathcal{M}, s \vDash \phi \wedge \psi & \Leftrightarrow & \mathcal{M}, s \vDash \phi \text{ and } \mathcal{M}, s \vDash \psi \\
\mathcal{M}, s \vDash \Box\psi & \Leftrightarrow & \forall t \rhd s : \mathcal{M}, t \vDash \psi \\
\mathcal{M}, s \vDash [\psi]\phi & \Leftrightarrow & \mathcal{M}, s \vDash \psi \text{ implies } \mathcal{M}|_\psi, s \vDash \phi
\end{array}$$

[1] For the simplicity of the exposition, we only consider the single agent case in this paper, all of our results and techniques apply to the multi-agent case as well.

where $(\forall t \rhd s : \ldots)$ denotes "for all $t : s \to t$ implies ...", and $\mathcal{M}|_\psi = (S', \to', V')$ such that: $S' = \{s \mid \mathcal{M}, s \vDash \psi\}$, $\to' = \to |_{S' \times S'}$ and $V'(p) = V(p) \cap S'$. According to this semantics, an announcement action $[\psi]$ is interpreted as a *model transformer* which deletes the worlds in the model that do not satisfy ψ.

In the literature, different Hilbert-style axiomatizations of PAL were proposed, see, e.g., Plaza 1989, Baltag and Moss (2004), van Benthem et al. (2006), and van Ditmarsch et al. (2007). Most of these axiomatizations are based on the following proof system **PA**:

System **PA**

Axiom schemata		Rules	
TAUT	All the instances of tautologies	MP	$\dfrac{\phi, \phi \to \psi}{\psi}$
DISTK	$\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$	NECK	$\dfrac{\phi}{\Box\phi}$
!ATOM	$[\psi]p \leftrightarrow (\psi \to p)$		
!NEG	$[\psi]\neg\phi \leftrightarrow (\psi \to \neg[\psi]\phi)$		
!CON	$[\psi](\phi \wedge \chi) \leftrightarrow ([\psi]\phi \wedge [\psi]\chi)$		
!K	$[\psi]\Box\phi \leftrightarrow (\psi \to \Box(\psi \to [\psi]\phi))$		

where ϕ, ψ, χ denote arbitrary formulas and $p \in \mathbf{P} \cup \{\top\}$. Note that **PA** does not include the rule of *uniform substitution* (US) since it is not valid, e.g., you cannot replace the p in !ATOM by an arbitrary formula. We will come back to this issue later on in our discussions.

In the epistemic setting, the corresponding axiomatization should also include the following **S5** axiom schemata:

$$\mathtt{T}: \Box\phi \to \phi \quad \mathtt{4}: \Box\phi \to \Box\Box\phi \quad \mathtt{5}: \neg\Box\phi \to \Box\neg\Box\phi$$

We abbreviate **PA**+T+4+5 as **PAK** in this paper.

1.2 Two basic questions

In the literature, several complete axiomitizations were proposed based on the above basic system **PA** with extra axioms or rules. However, in many published works, **PA** and **PAK** are also mentioned as "complete" axiomatizations of PAL without a proof. Is it correct? The first task of this paper is to give a definite answer to the following question:

Question 1: *Are* **PA** *and* **PAK** *complete w.r.t. the corresponding frame classes?*

In this paper, we will actually examine many additional axiom schemata and rules mentioned in the literature to see whether they are necessary in a complete axiomitization. Here are the additional axiom schemata and rules that we will discuss in this paper:

Additional axiom schemata and rules			
Axiom schemata		Rules	
DIST!	$[\psi](\phi \to \chi) \to ([\psi]\phi \to [\psi]\chi)$	NEC!	$\dfrac{\phi}{[\psi]\phi}$
!COMP	$[\psi][\chi]\phi \leftrightarrow [\psi \wedge [\psi]\chi]\phi$	RE	$\dfrac{\phi \leftrightarrow \chi}{\psi \leftrightarrow \psi[\chi/\phi]}$
WDIST!	$[\psi](\phi \to \chi) \leftrightarrow ([\psi]\phi \to (\psi \to [\psi]\chi))$	RE $\neg$	$\dfrac{\phi \leftrightarrow \chi}{\neg\phi \leftrightarrow \neg\chi}$
SDIST!	$[\psi](\phi \to \chi) \leftrightarrow ([\psi]\phi \to [\psi]\chi)$	RE$\wedge$	$\dfrac{\phi \leftrightarrow \chi}{(\psi \wedge \phi) \leftrightarrow (\psi \wedge \chi)}$
!K′	$[\psi]\Box\phi \leftrightarrow (\psi \to \Box[\psi]\phi)$	RE$\Box$	$\dfrac{\phi \leftrightarrow \chi}{\Box\phi \leftrightarrow \Box\chi}$
PRE	$(\psi \to [\psi]\phi) \to [\psi]\phi$	RE!	$\dfrac{\phi \leftrightarrow \chi}{[\psi]\phi \leftrightarrow [\psi]\chi}$
PFUNC	$(\psi \wedge [\psi]\phi) \leftrightarrow \langle\psi\rangle\phi$	!RE	$\dfrac{\phi \leftrightarrow \chi}{[\phi]\psi \leftrightarrow [\chi]\psi}$

The selection of these axioms and rules is not arbitrary: RE is the rule of *replacement of equivalents* where $\psi[\chi/\phi]$ denotes any formula obtained by replacing one or more *non-modality* occurrences of ϕ in ψ with χ. Here non-modality occurrences of ϕ are the occurrences of ϕ which are *not* inside any [ ].[2] !COMP is the *composition axiom* explicitly proposed in van Ditmarsch (2003) and it is used in many later expositions of PAL (e.g., van Ditmarsch et al. 2007).[3] RE$\neg$, RE$\wedge$, RE$\Box$, RE! are weaker versions of RE, and !RE handles the replacement inside the announcements. !K′ is often used in the literature as an "equivalent" of !K (cf., e.g., van Benthem et al. 2006), while NEC! and DIST! are $[\phi]$-versions of the well-known necessitation rule and distribution axiom in basic modal logic, which sometimes appear too in the axiomatizations of PAL (cf. e.g. Baltag and Moss 2004). WDIST! and SDIST! are (weaker/stronger) variations of DIST! that we will use as auxillary axioms for the proofs. Finally PRE and PFUNC are often taken for granted in the previous works based on **PA**. PRE says that ψ is the precondition of $[\psi]\phi$ and PFUNC is the direct definition of $\langle\psi\rangle\phi$.

Note that besides the usual EL axioms and rules, **PA** features a set of so-called *reduction axioms* (!ATOM, !NEG, !CON and !K). Reading from the left-hand-side to the right-hand-side, such axioms can be seen as truth-preserving rewriting rules which push the announcement modalities to the inner part of the formula. Eventually we may eliminate the announcement operators (see the shape of !ATOM). As we will review in the next section, with the help of extra axioms and rules besides the ones in **PA**, we can show that PAL formulas are provably equivalent to EL formulas. The completeness of such a reduction-based axiomatization of PAL can then be

[2] Note that the usual rule of replacement of logical equivalents as in Plaza (1989) and many other works is stronger than our RE in the sense that it is not restricted to the non-modality occurrences and can be viewed as a combination of our RE and !RE rules. The separation of RE and !RE helps us to pinpoint exactly the rules that are needed to make **PA** complete.

[3] The corresponding composition phenomenon was observed earlier by van Benthem (1999) as the associativity of syntactic relativization. A more general version of this composition axiom in the setting of DEL with event models was first mentioned in Baltag et al. (1998).

reduced to the completeness of EL systems. This reduction technique has proven to be extremely useful in DEL (cf. e.g., Baltag and Moss 2004; van Benthem et al. 2006). Moreover, as van Benthem (1999) pointed out, the announcement update can be viewed as a *semantic relativization* $(\cdot)|_\psi$ operator turning a model into a definable submodel, while the reduction axioms corresponds to the recursive definition of the *syntactic relativisation* $(\cdot)^\psi$ such that the semantic relativization can be turned into syntactic relativization in the following sense (cf. van Benthem 2011, Chap. 3.12)[4]:

$$\mathcal{M}|_\psi, s \vDash \phi \iff \mathcal{M}, s \vDash \phi^\psi.$$

It is also very interesting to examine the logics extending epistemic logic to see whether they have enough *encoding power* to obtain the reduction for various updates, see, e.g., van Benthem et al. (2006) and van Benthem and Ikegami (2008).

However, are the reduction axioms the only meaningful axioms to characterize PAL? What if there is no such reduction possible?[5] Thus our second task in this paper is to answer the following question:

Question 2: *Is it possible to give a meaningful axiomatization of* PAL *without using reduction axioms and the reduction technique in the completeness proof?*

Actually, such an attempt was made in one of the earliest works on PAL by Gerbrandy and Groeneveld (1997). At that time, Gerbrandy and Groeneveld were not aware of the reduction-style axiomitization as in Plaza (1989), therefore they first proposed their own set of axioms for a variation of PAL language w.r.t a different semantics based on non-well-founded sets. They also proved the completeness of their system by using the canonical model method. This approach was abandoned in Gerbrandy (1999),[6] where the reduction axioms were rediscovered independently. It seems to us that the merit of Gerbrandy and Groeneveld's earlier approach has been largely forgotten and the reduction method has become the key technique in the field of DEL.[7]

In this paper, we propose an alternative yet meaningful axiomatization of PAL, similar to the one given by Gerbrandy and Groeneveld (1997), as follows:

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[4] Due to the connection with the recursive definition of the syntactic relativization, van Benthem (2011) advocates the name "*recursion axioms*" than the reduction axioms, since the reduction may not be the main goal. In this paper we stick to the usual name.

[5] PALwith common knowledge operator is such an example (van Ditmarsch et al. 2007).

[6] Gerbrandy (1999) mentioned that he has to abandon NEC! in order to cope with private updates which do not preserve **S5** frame properties. Therefore the usual canonical model method does not work any more in the absence of NEC!.

[7] Here we want to note that there is also a significant body of research going beyond the "orthodox" reduction programme of DEL. As an example, see van Benthem (2011, Chap. 11)and the references therein. We will come back to some of such works in Sect. 5 in detail.

System **PAN**			
Axiom schemata		Rules	
TAUT	All the instances of tautologies	MP	$\dfrac{\phi, \phi \to \psi}{\psi}$
DISTK	$\Box(\phi \to \chi) \to (\Box\phi \to \Box\chi)$	NECK	$\dfrac{\phi}{\Box\phi}$
DIST!	$[\psi](\phi \to \chi) \to ([\psi]\phi \to [\psi]\chi)$	NEC!	$\dfrac{\phi}{[\psi]\phi}$
INV	$(p \to [\psi]p) \wedge (\neg p \to [\psi]\neg p)$		
PFUNC	$\langle\psi\rangle\phi \leftrightarrow (\psi \wedge [\psi]\phi)$		
NM	$\Diamond\langle\psi\rangle\phi \to [\psi]\Diamond\phi$		
PR	$\langle\psi\rangle\Diamond\phi \to \Diamond\langle\psi\rangle\phi$		

The new axiomatization makes use of the axiom schemata of *perfect recall* (PR), *no miracles* (NM) and two other axiom schemata which illustrate the following features of the updates: *partial-functionality* (PFUNC) and *propositional invariance* (INV).

The readers may wonder whether our axiom schemata are merely some reshuffles of the reduction axiom schemata. In fact, as we shall see later, they do have deep roots in our completeness proof method that accompanies the axiomitization. Our proof method does not use any reduction to epistemic logic and can be useful to other dynamic epistemic logics even when reduction is not possible. Moreover, **PAN** may sharpen our understanding of PAL and DEL in general, as we will discuss in Sect. 5. In particular, we will relate **PAN** to some recent results in the field, including the axiomatization of the "substitution core" of PAL in Holliday et al. (2012), the axiomatization of the substitution-closed epistemic action logic in Wang and Li (2012), the characterization of partial p-morphism using reduction axioms in van Benthem (2012), and the representation theorems of DEL in van Benthem et al. (2009) and Dégremont et al. (2011). These analyses may shed new light on the essence of PAL and other dynamic epistemic logics.

The main technical contributions of this paper can be summarized as follows (note that all the results still hold if we replace **PA** by **PAK** and consider the completeness w.r.t. **S5** frames):

- **PA**, **PA** + DIST! and **PA** + NEC! are *not* complete, and RE is *not* an admissible rule of **PA**.[8] A detailed summary of results concerning other axioms is provided at the end of Sect. 3.
- The system **PAN** is sound and complete. The completeness is proved by a detour method using the canonical model where announcements are treated as merely labels for modalities as in normal modal logics. This method can be applied to other dynamic epistemic logics.

[8] Here we say an inference rule $\dfrac{\phi}{\psi}$ is *derivable* from a system **S** if ψ can be derived by using ϕ, the axiom schemata and inference rules of **S**. An inference rule is *admissible* in **S** if the set of theorems stays the same when this rule is added to **S**. Given a system, a derivable rule is clearly admissible but an admissible rule may not be derivable.

The rest of this paper is organized as follows: In Sect. 2, we review the most well-known axiomatizations of **PA** and make some useful observations about additional axioms and rules. In Sect. 3, by giving two non-standard semantics which validate **PA** but invalidate some of the valid formulas and rules, we show that **PA** and some of its extensions are not complete w.r.t. the standard PAL semantics. We propose a general proof strategy and use it to prove the completeness of **PAN** in Sect. 4, without using the reduction to epistemic logic. We devote Sect. 5 to a very detailed discussion of the axioms in **PAN** and many related results in the recent literature of the field. Finally, we conclude with future work in Sect. 6.

2 Preliminaries

Proposition 1 **PA** *is sound w.r.t to the standard* PAL *semantics.*

Proof Cf. e.g., van Ditmarsch et al. (2007). □

Moreover, it is an easy exercise to show that all the other axioms and rules mentioned in the introduction are valid w.r.t. to the standard PAL semantics:

Proposition 2 *Axiom schemata* DIST!, !COMP, WDIST!, SDIST!, !K′, PRE, FUNC, *and inference rules* NEC!, RE, RE¬, RE∧, RE□, RE!, !RE *are all valid w.r.t. the standard* PAL *semantics.*

A natural question to ask is: are they derivable in **PA**? We list a few positive answers here.

Proposition 3 RE¬, RE∧, RE□ *can be derived in* **PA**. *On the other hand,* RE! *can be derived in* **PA**+NEC!+DIST!.

Proof RE¬, RE∧ are trivial by using TAUT. Here we only show the (standard) reasoning behind RE□.

$1 \vdash_{\mathbf{PA}} \phi \leftrightarrow \chi$	
$2 \vdash_{\mathbf{PA}} \phi \to \chi$	TAUT
$3 \vdash_{\mathbf{PA}} \Box(\phi \to \chi)$	NECK
$4 \vdash_{\mathbf{PA}} \Box(\phi \to \chi) \to (\Box\phi \to \Box\chi)$	DISTK
$5 \vdash_{\mathbf{PA}} \Box\phi \to \Box\chi$	MP(3, 4)
$6 \vdash_{\mathbf{PA}} \Box\phi \leftrightarrow \Box\chi$	repeat 2–5 for $\chi \to \phi$, TAUT

Note that the above proof uses NECK and DISTK. Similarly we can prove RE! in **PA**+NEC!+DIST!. However, as we will see in Sect. 3, DIST! and NEC! cannot be derived in **PA**. □

Based on the above proposition, we know that the following restricted version of RE holds.

Proposition 4 *The following rule* RE^r *is admissible in* **PA**: *Given* $\phi \leftrightarrow \chi$, *we have* $\psi \leftrightarrow \psi'$ *where* ψ' *is obtained by replacing some non-modality occurrences of* ϕ *in* ψ *with* χ, *provided that these occurrences of* ϕ *are also "announcement-irrelevant", i.e., they do not appear in the scope of any announcement operator.*

Proof Suppose $\vdash_{\mathbf{PA}} \phi \leftrightarrow \chi$ and ψ' is obtained from ψ by replacing some announcement-irrelevant occurrences of ϕ by χ. It is not hard to see that we can construct ψ and ψ' from ϕ, χ and other formulas by using the equivalence preserving operations denoted by $\mathtt{RE}\neg$, $\mathtt{RE}\wedge$, $\mathtt{RE}\Box$. Note that it does *not* mean we can only handle announcement-free formulas, i.e., from $\vdash_{\mathbf{PA}} \phi \leftrightarrow \chi$ we can show that $\vdash_{\mathbf{PA}} ([\psi]\phi \to \Box\phi) \leftrightarrow ([\psi]\phi \to \Box\chi)$ by taking $[\psi]\phi$ as one of the atomic building blocks. □

Since RE! is derivable in **PA**+NEC!+DIST!, we can show the following proposition based on a similar proof like the above.

Proposition 5 *The rule* RE *is admissible in* **PA**+NEC!+DIST!.

Now let us look at other axiom schemata.

Proposition 6 PFUNC *is a theorem schema of* **PA**.

Proof

1	$\vdash_{\mathbf{PA}} [\psi]\neg\phi \leftrightarrow (\psi \to \neg[\psi]\phi)$	!NEG
2	$\vdash_{\mathbf{PA}} \neg[\psi]\neg\phi \leftrightarrow \neg(\psi \to \neg[\psi]\phi)$	RE¬(1)
3	$\vdash_{\mathbf{PA}} \neg(\psi \to \neg[\psi]\phi) \leftrightarrow (\psi \wedge [\psi]\phi)$	TAUT
4	$\vdash_{\mathbf{PA}} \neg[\psi]\neg\phi \leftrightarrow (\psi \wedge [\psi]\phi)$	$\mathtt{RE}^r$(2, 3)

□

Proposition 7 WDIST! *is a theorem schema of* **PA**.

Proof Note that $\phi \to \chi$ is the abbreviation of $\neg(\phi \wedge \neg\chi)$. Thus $[\psi](\phi \to \chi)$ is the abbreviation of $[\psi]\neg(\phi \wedge \neg\chi)$.

1	$\vdash_{\mathbf{PA}} [\psi]\neg(\phi \wedge \neg\chi) \leftrightarrow (\psi \to \neg[\psi](\phi \wedge \neg\chi))$	!NEG
2	$\vdash_{\mathbf{PA}} [\psi](\phi \wedge \neg\chi) \leftrightarrow ([\psi]\phi \wedge [\psi]\neg\chi)$	!CON
3	$\vdash_{\mathbf{PA}} \neg[\psi](\phi \wedge \neg\chi) \leftrightarrow \neg([\psi]\phi \wedge [\psi]\neg\chi)$	RE¬(2)
4	$\vdash_{\mathbf{PA}} [\psi]\neg\chi \leftrightarrow (\psi \to \neg[\psi]\chi)$	!NEG
5	$\vdash_{\mathbf{PA}} \neg[\psi](\phi \wedge \neg\chi) \leftrightarrow \neg([\psi]\phi \wedge (\psi \to \neg[\psi]\chi))$	$\mathtt{RE}^r$(3, 4)
6	$\vdash_{\mathbf{PA}} [\psi](\phi \to \chi) \leftrightarrow (\psi \to \neg([\psi]\phi \wedge (\psi \to \neg[\psi]\chi))$	$\mathtt{RE}^r$(5, 1)
7	$\vdash_{\mathbf{PA}} [\psi](\phi \to \chi) \leftrightarrow (\psi \to ([\psi]\phi \to (\psi \wedge [\psi]\chi)))$	TAUT
8	$\vdash_{\mathbf{PA}} [\psi](\phi \to \chi) \leftrightarrow ((\psi \wedge [\psi]\phi) \to (\psi \wedge [\psi]\chi))$	TAUT
9	$\vdash_{\mathbf{PA}} [\psi](\phi \to \chi) \leftrightarrow ((\psi \wedge [\psi]\phi) \to [\psi]\chi)$	TAUT
10	$\vdash_{\mathbf{PA}} [\psi](\phi \to \chi) \leftrightarrow ([\psi]\phi \to (\psi \to [\psi]\chi))$	TAUT

□

Note that $\vdash_{\mathbf{PA}} [\psi]\chi \to (\psi \to [\psi]\chi)$, thus if $\vdash_{\mathbf{PA}}$ PRE then $\vdash_{\mathbf{PA}} [\psi]\chi \leftrightarrow (\psi \to [\psi]\chi)$. Now since WDIST! and $\mathtt{RE}^r$ are derivable in **PA**, it is clear that if $\vdash_{\mathbf{PA}}$ PRE then SDIST! (and DIST!) can be proved in **PA**. However, $\nvdash_{\mathbf{PA}}$ PRE as we will see in Sect. 3.

If we extend **PA** with PRE and NEC!, then RE! is derivable.

Proposition 8 RE! *is derivable in* **PA**+PRE+NEC! *and* **PA**+DIST!+NEC!. *Therefore* RE *is admissible in* **PA**+PRE+NEC! *and* **PA**+DIST!+NEC!.

Proof Note that $\vdash_{\mathbf{PA}+\mathtt{PRE}}$ DIST!. Then with NEC! we can derive RE! (cf. the proof of Proposition 3). □

Proposition 9 PRE *is a theorem schema of* **PA**+!COMP.

Proof By induction on the structure of ϕ (van Ditmarsch et al. 2007, p. 251). □

By using the reduction axioms in **PA** and the above restricted substitution rule we can translate most of PAL formulas to equivalent EL formulas by iteratively replacing the inner part of the formula with an equivalent announcement-free formula. However, formulas in the shape of $[\psi][\chi]\phi$ may be problematic since RE! is missing in **PA**.

Here we mention a few completeness results by using reductions.

Theorem 10 **PA**+!COMP *is sound and (weakly) complete w.r.t. the standard semantics of* PAL.

Proof We only sketch the proof in van Ditmarsch et al. (2007).[9] We first define a translation t : PAL → EL as follows:

$$
\begin{array}{llll}
t(\top) & = \top & t([\psi]\top) & = t(\psi \to \top) \\
t(p) & = p & t([\psi]p) & = t(\psi \to p) \\
t(\neg\phi) & = \neg t(\phi) & t([\psi]\neg\phi) & = t(\psi \to \neg[\psi]\phi) \\
t(\phi_1 \wedge \phi_2) & = t(\phi_1) \wedge t(\phi_2) & t([\psi](\phi_1 \wedge \phi_2)) & = t([\psi]\phi_1 \wedge [\psi]\phi_2) \\
t(\Box\phi) & = \Box t(\phi) & t([\psi]\Box\phi) & = t(\psi \to \Box(\psi \to [\psi]\phi)) \\
 & & t([\psi][\chi]\phi) & = t([\psi \wedge [\psi]\chi]\phi)
\end{array}
$$

Based on a suitable definition of the complexity of formulas (cf. van Ditmarsch et al. 2007) we can show that the translation/rewriting always reduces the complexity thus it will terminate at some point. Note that in the process of the rewriting, $t(\psi)$ never falls in the scope of any announcement operator. Based on this observation, by induction on the complexity of the formulas we can show that $\vdash_{\mathbf{PA}+!\mathtt{COMP}} \phi \leftrightarrow t(\phi)$ (using reduction axioms, !COMP, RE∧, RE¬, and RE□). By soundness of **PA**+!COMP, we have $\vDash \phi \leftrightarrow t(\phi)$. Now suppose $\vDash \phi$ then $\vDash t(\phi)$. Thus by the completeness of the basic modal logic **K**, $\vdash_{\mathbf{K}} t(\phi)$. Therefore $\vdash_{\mathbf{PA}+!\mathtt{COMP}} t(\phi)$ since **PA** includes all the axioms and rules of **K**. Since $\vdash_{\mathbf{PA}+!\mathtt{COMP}} \phi \leftrightarrow t(\phi)$, we have $\vdash_{\mathbf{PA}+!\mathtt{COMP}} \phi$ by MP. □

Theorem 11 (Plaza 1989) **PA**+ RE *is sound and (weakly complete) w.r.t. the standard semantics of* PAL.

Proof Similar to the above proof, we only need to revise the last item of the translation function t as follows:

$$t([\psi][\chi]\phi) = t([\psi]t([\chi]\phi))$$

Note that now we do need the full power of RE since t does fall in the scope of announcement operators. □

[9] We need to adapt the proof just a little bit to fit !K in the proof instead of !K′ used in van Ditmarsch et al. (2007).

As a straightforward corollary, we have:

Corollary 12 **PA**+DIST!+NEC!, **PA**−!CON+DIST!+NEC! *and* **PA**+PRE+NEC! *are sound and complete w.r.t. the standard semantics of* PAL.

Proof From Proposition 8, the above theorem and the fact that !CON is derivable from NEC! and DIST! as an easy exercise. □

Remark 1 The translation of $[\psi][\chi]\phi$ formulas defined in Theorem 11 is in the fashion of "inside-out" while the translation in Theorem 10 is "outside-in".

3 PA is not complete

To show that a formula ϕ is not derivable in a system **S**, a usual strategy is to design a semantics such that **S** is sound w.r.t. this semantics but ϕ is not valid w.r.t. this semantics. In this section, we give two alternative semantics for the language of PAL which validate **PA** but make many intuitive axioms and rules invalid.

3.1 A context-dependent semantics

Inspired by the semantics developed in Gabbay (2002), Wang (2006), and Bonnay and Égré (2009), we define the satisfaction relation w.r.t. a *context* ρ (notation: $\vDash_\rho$), which is used to record the information from the previous announcements.

Given a Kripke model over **P**: $\mathcal{M} = (S, \to, V)$, the truth value of a PAL formula ϕ at a state s in $\mathcal{M}$ is recursively defined as follows based on a context-dependent satisfaction relation $\Vdash_\rho$ where ρ is a formula in the language of PAL:

$$
\begin{array}{rcl}
\mathcal{M}, s \Vdash \phi & \Leftrightarrow & \mathcal{M}, s \Vdash_\top \phi \\
\mathcal{M}, s \Vdash_\rho \top & \Leftrightarrow & \text{always} \\
\mathcal{M}, s \Vdash_\rho p & \Leftrightarrow & s \in V(p) \\
\mathcal{M}, s \Vdash_\rho \neg\phi & \Leftrightarrow & \mathcal{M}, s \nVdash_\rho \phi \\
\mathcal{M}, s \Vdash_\rho \phi \wedge \psi & \Leftrightarrow & \mathcal{M}, s \Vdash_\rho \phi \text{ and } \mathcal{M}, s \Vdash_\rho \psi \\
\mathcal{M}, s \Vdash_\rho \Box\phi & \Leftrightarrow & \forall t \rhd s : \mathcal{M}, t \Vdash_\top \rho \text{ implies } \mathcal{M}, t \Vdash_\rho \phi \\
\mathcal{M}, s \Vdash_\rho [\psi]\phi & \Leftrightarrow & \mathcal{M}, s \Vdash_\top \psi \text{ implies } \mathcal{M}, s \Vdash_{\rho\wedge\psi} \phi
\end{array}
$$

Note that instead of updating the model we remember the announcements and recall them only in evaluating $\Box$ formulas. Remembering the context is an alternative way of doing model relativization. We will come back to this idea in Sect. 6 at the end of the paper. As usual, we say that ϕ is *valid* ($\Vdash \phi$) w.r.t. this non-standard semantics if for all the pointed model $\mathcal{M}, s$: $\mathcal{M}, s \Vdash \phi$ (i.e. $\mathcal{M}, s \Vdash_\top \phi$).

Example 1 Consider the following (**S5**) model $\mathcal{M}$ with two worlds s, v:

$$
\begin{tikzcd}
s : p \arrow[loop above, "i"] \arrow[r, leftrightarrow, "i"] & v : \neg p \arrow[loop above, "i"]
\end{tikzcd}
$$

$$\mathcal{M}, s \Vdash \neg\Box p \iff \mathcal{M}, s \nVdash_\top \Box p \iff (\exists t \rhd s : \mathcal{M}, t \Vdash_\top \top \text{ and } \mathcal{M}, t \nVdash_\top p)$$
Since $v \notin V(p)$ and $s \xrightarrow{i} v$, $\mathcal{M}, s \Vdash \neg\Box p$.

$\mathcal{M}, s \Vdash_p \Box p \iff (\forall t \rhd s : \mathcal{M}, t \Vdash_\top p$ implies $\mathcal{M}, t \Vdash_p p)$. Clearly, $\mathcal{M}, s \Vdash_p \Box p$. Similarly $\mathcal{M}, s \Vdash_{\top\wedge p} \Box p$.

$\mathcal{M}, s \Vdash [p]\Box p \iff (\mathcal{M}, s \Vdash_\top p$ implies $\mathcal{M}, s \Vdash_{\top\wedge p} \Box p)$
$\iff \mathcal{M}, s \Vdash_{\top\wedge p} \Box p$. Thus, $\mathcal{M}, s \Vdash [p]\Box p$ (based on the above example).

$\mathcal{M}, s \Vdash [p\wedge\neg\Box p]\Box p \iff (\mathcal{M}, s \Vdash_\top p\wedge\neg\Box p$ implies $\mathcal{M}, s \Vdash_{\top\wedge p\wedge\neg\Box p} \Box p)$
$\iff (\mathcal{M}, s \Vdash_\top p$ and $\mathcal{M}, s \Vdash_\top \neg\Box p)$ implies $\mathcal{M}, s \Vdash_{\top\wedge p\wedge\neg\Box p} \Box p$
$\iff \mathcal{M}, s \Vdash_{\top\wedge p\wedge\neg\Box p} \Box p$ (from the above examples)
$\iff \forall t \rhd s : \mathcal{M}, t \Vdash_\top \top \wedge p \wedge \neg\Box p$ implies $\mathcal{M}, t \Vdash_{\top\wedge p\wedge\neg\Box p} p$.

It is easy to see that $\mathcal{M}, s \Vdash [p \wedge \neg\Box p]\Box p$.

$\mathcal{M}, s \Vdash [p][\neg\Box p]\bot \iff (\mathcal{M}, s \Vdash_\top p$implies $\mathcal{M}, s \Vdash_{\top\wedge p} [\neg\Box p]\bot)$
$\iff \mathcal{M}, s \Vdash_\top \neg\Box p$ implies $\mathcal{M}, s \Vdash_{\top\wedge p\wedge\neg\Box p} \bot$. Thus $\mathcal{M}, s \nVdash [p][\neg\Box p]\bot$.

On the other hand, it is easy to verify that $\mathcal{M}, s \vDash [p][\neg\Box p]\bot$ (recall that $\vDash$ denotes the standard semantics).

In the above example, it seems that $\Vdash$ coincides with $\vDash$ except for the formulas with consecutive announcements. We will show that it is not a coincidence.

Proposition 13 $\Vdash$ *coincides with* $\vDash$ *on* EL *formulas.*

Proof Note that without $[\psi]$ operators, ρ can never be changed to any non-trivial formula during the evaluation of a formula. Since $\mathcal{M}, s \Vdash_\top \top$ is always true, it is easy to see that the definition of $\Vdash_\top$ coincides with $\vDash$ for EL formulas. $\Box$

Before going further we first prove two useful propositions. Let !COMP$\wedge$ be the axiom schema $[\psi][\chi]\phi \leftrightarrow [\psi \wedge \chi]\phi$ which is different from !COMP.

Proposition 14 !COMP$\wedge$ *is valid w.r.t.* $\Vdash$.

Proof $\mathcal{M}, s \Vdash [\psi][\chi]\phi \iff \mathcal{M}, s \Vdash_\top \psi$ implies $\mathcal{M}, s \Vdash_{\top\wedge\psi} [\chi]\phi$

$$\iff \mathcal{M}, s \Vdash_\top \psi \text{ implies } (\mathcal{M}, s \Vdash_\top \chi \text{ implies } \mathcal{M}, s \Vdash_{\top\wedge\psi\wedge\chi} \phi)$$

$$\iff (\mathcal{M}, s \Vdash_\top \psi \text{ and } \mathcal{M}, s \Vdash_\top \chi) \text{ implies } \mathcal{M}, s \Vdash_{\top\wedge\psi\wedge\chi} \phi$$

$$\iff (\mathcal{M}, s \Vdash_\top \psi \wedge \chi) \text{ implies } \mathcal{M}, s \Vdash_{\top\wedge\psi\wedge\chi} \phi$$

$$\iff \mathcal{M}, s \Vdash [\psi \wedge \chi]\phi$$

$\Box$

Proposition 15 *For any* PAL *formulas* χ, ψ, *and* ϕ : *if* $\Vdash \chi \leftrightarrow \psi$ *then for all pointed model* $\mathcal{M}, s$: $\mathcal{M}, s \Vdash_\chi \phi \iff \mathcal{M}, s \Vdash_\psi \phi$. *As a consequence,* !RE *is valid w.r.t.* $\Vdash$.

Proof First note that !RE is not RE!. By induction on the structure of ϕ. The Boolean cases are trivial. Now let $\phi = \Box\phi'$. Note that $\mathcal{M}, s \Vdash_\rho \Box\phi' \iff \forall t \rhd s : \mathcal{M}, t \Vdash_\top \rho$ implies $\mathcal{M}, t \Vdash_\rho \phi'$. Since $\Vdash \chi \leftrightarrow \psi$, for all $\mathcal{M}, t$: $\mathcal{M}, t \Vdash_\top \chi \iff \mathcal{M}, t \Vdash_\top \psi$. Therefore, based on the induction hypothesis that $\mathcal{M}, t \Vdash_\chi \phi' \iff \mathcal{M}, t \Vdash_\psi \phi'$, $\mathcal{M}, s \Vdash_\chi \Box\phi \iff \mathcal{M}, s \Vdash_\psi \Box\phi$.

Now consider $\phi = [\phi']\phi''$. According to the semantics of conjunctions, it is not hard to see that if $\Vdash \psi \leftrightarrow \chi$ then for any ϕ' we have $\Vdash (\psi \wedge \phi') \leftrightarrow (\chi \wedge \phi')$.

Now according to the truth condition of $[\phi']\phi''$ and induction hypothesis, $\mathcal{M}, s \Vdash_\chi [\phi']\phi'' \iff \mathcal{M}, s \Vdash_\psi [\phi']\phi''$. Based on the these observations, it is easy to show that $\Vdash \psi \leftrightarrow \chi$ implies $\Vdash [\chi]\phi \leftrightarrow [\psi]\phi$. □

Remark 2 The inference rule !RE is itself interesting in axiomatizing PAL. By inductive proofs using reduction axioms, it is not hard to show that !RE is admissible in both **PA**+RE and **PA**+!COMP. We conjecture that it is not admissible in **PA** but leave it for future work.

In the following we show that **PA** is sound w.r.t. $\Vdash$. Actually, many other rules and axiom schemata are also valid under $\Vdash$ as we will see soon.

Lemma 16 TAUT, MP, NECK, *and* DISTK *are valid w.r.t.* $\Vdash$.

Proof For TAUT and MP: Trivial (check the truth conditions for Boolean cases).

For NECK: Suppose $\Vdash \phi$ then for all models $\mathcal{M}, s \Vdash_\top \phi$. Suppose towards a contradiction that there is a model $\mathcal{M}, s \Vdash_\top \neg\Box\phi$. According to the semantics there exists $t \rhd s$ $\mathcal{M}, t \Vdash_\top \top$ and $\mathcal{M}, t \Vdash_{\top\wedge\top} \neg\phi$, contradiction.

For DISTK: Suppose $\mathcal{M}, s \Vdash \Box(\phi \to \psi)$ then for all $t \rhd s$ $\mathcal{M}, t \Vdash_\top \phi \to \psi$. Now suppose $\mathcal{M}, s \Vdash \Box\phi$ then for all $t \rhd s$: $\mathcal{M}, t \Vdash_\top \top$ implies $\mathcal{M}, t \Vdash_\top \phi$. It is clear that for all $t \rhd s$: $\mathcal{M}, t \Vdash_\top \top$ implies $\mathcal{M}, t \Vdash_\top \psi$. Thus $\mathcal{M}, s \Vdash \Box\psi$. Therefore $\mathcal{M}, s \Vdash \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$. □

Lemma 17 !ATOM, !NEG, !CON, !K, !K$'$, *and* PRE *are valid w.r.t.* $\Vdash$.

Proof For !ATOM: $\mathcal{M}, s \Vdash [\psi]p \iff (\mathcal{M}, s \Vdash_\top \psi$ implies $\mathcal{M}, s \Vdash_{\top\wedge\psi} p)$

$\iff (\mathcal{M}, s \Vdash_\top \psi$ implies $\mathcal{M}, s \Vdash_\top p) \iff \mathcal{M}, s \Vdash \psi \to p$.

For !NEG: $\mathcal{M}, s \Vdash [\psi]\neg\phi \iff (\mathcal{M}, s \Vdash_\top \psi$ implies $\mathcal{M}, s \Vdash_{\top\wedge\psi} \neg\phi)$ while $\mathcal{M}, s \Vdash \psi \to \neg[\psi]\phi \iff (\mathcal{M}, s \Vdash_\top \psi$ implies $\mathcal{M}, s \Vdash_\top \neg[\psi]\phi)$

$\iff \mathcal{M}, s \Vdash_\top \psi$ implies $(\mathcal{M}, s \Vdash_\top \psi$ and $\mathcal{M}, s \Vdash_{\top\wedge\psi} \neg\phi)$
$\iff \mathcal{M}, s \Vdash_\top \psi$ implies $\mathcal{M}, s \Vdash_{\top\wedge\psi} \neg\phi$. Thus $\mathcal{M}, s \Vdash [\psi]\neg\phi \leftrightarrow (\psi \to \neg[\psi]\phi)$.

For !CON: $\mathcal{M}, s \Vdash [\psi](\phi \wedge \chi) \iff (\mathcal{M}, s \Vdash_\top \psi$ implies $\mathcal{M}, s \Vdash_{\top\wedge\psi} \phi \wedge \chi)$
$\iff (\mathcal{M}, s \Vdash_\top \psi$ implies $\mathcal{M}, s \Vdash_{\top\wedge\psi} \phi)$ and $(\mathcal{M}, s \Vdash_\top \psi$ implies $\mathcal{M}, s \Vdash_{\top\wedge\psi} \chi) \iff \mathcal{M}, s \Vdash_\top [\psi]\phi \wedge [\psi]\chi$.

For !K: $\mathcal{M}, s \Vdash [\psi]\Box\phi \iff \mathcal{M}, s \Vdash_\top \psi$ implies $\mathcal{M}, s \Vdash_{\top\wedge\psi} \Box\phi$ while $\mathcal{M}, s \Vdash \psi \to \Box(\psi \to [\psi]\phi) \iff \mathcal{M}, s \Vdash_\top \psi$ implies $\mathcal{M}, s \Vdash_\top \Box(\psi \to [\psi]\phi)$

$\iff \mathcal{M}, s \Vdash_\top \psi$ implies $(\forall t \rhd s : \mathcal{M}, t \Vdash_\top \top$ implies $(\mathcal{M}, t \Vdash_\top \psi$ implies $\mathcal{M}, t \Vdash_\top [\psi]\phi))$
$\iff \mathcal{M}, s \Vdash_\top \psi$ implies $(\forall t \rhd s : \mathcal{M}, t \Vdash_\top \psi$ implies $\mathcal{M}, t \Vdash_\top [\psi]\phi)$
$\iff \mathcal{M}, s \Vdash_\top \psi$ implies $(\forall t \rhd s : \mathcal{M}, t \Vdash_\top \psi$ implies $(\mathcal{M}, t \Vdash_\top \psi$ implies $\mathcal{M}, t \Vdash_{\top\wedge\psi} \phi))$
$\iff \mathcal{M}, s \Vdash_\top \psi$ implies $(\forall t \rhd s : \mathcal{M}, t \Vdash_\top \psi$ implies $\mathcal{M}, t \Vdash_{\top\wedge\psi} \phi)$
$\iff \mathcal{M}, s \Vdash_\top \psi$ implies $(\forall t \rhd s : \mathcal{M}, t \Vdash_\top \top \wedge \psi$ implies $\mathcal{M}, t \Vdash_{\top\wedge\psi} \phi)$
$\iff \mathcal{M}, s \Vdash_\top \psi$ implies $\mathcal{M}, s \Vdash_{\top\wedge\psi} \Box\phi$

Thus $\mathcal{M}, s \Vdash [\psi]\Box\phi \leftrightarrow (\psi \to \Box(\psi \to [\psi]\phi))$. Similarly, we can verify that $!\mathtt{K}'$ is valid w.r.t. $\Vdash$.

For PRE: immediate from the implication form of the truth condition of $[\psi]\phi$. □

Based on the Proposition 15, Lemmata 17 and 16, we can prove the soundness of **PA** and some of its extensions w.r.t. $\Vdash$.

Theorem 18 *For all* PAL *formulas* ϕ: $\vdash_{\mathbf{PA}+\mathtt{PRE}+!\mathtt{K}'+!\mathtt{RE}} \phi$ *implies* $\Vdash \phi$.

Now we prove that many axioms and rules we mentioned in the introduction are *not* derivable in **PA**, by showing that they are not valid w.r.t $\Vdash$.

Lemma 19 *None of* !COMP, NEC!, RE!, *and* RE *is valid under* $\Vdash$.

Proof For !COMP: We consider $[p][\Box p]\bot$ and $[p \wedge [p]\Box p]\bot$. From Proposition 14,

$$\Vdash [p][\Box p]\bot \leftrightarrow [p \wedge \Box p]\bot.$$

Note that $[p]\Box p$ is valid w.r.t. $\Vdash$ thus $\Vdash [p]\Box p \leftrightarrow \top$. From Proposition 15, $\Vdash [p \wedge [p]\Box p]\bot \leftrightarrow [p \wedge \top]\bot \leftrightarrow [p]\bot$. However, $[p \wedge \Box p]\bot \leftrightarrow [p]\bot$ is not valid e.g., on the following (**S5**) model:

$$\begin{tikzcd} s : p \arrow[loop above, "i"] \arrow[r, leftrightarrow, "i"] & v : \neg p \arrow[loop above, "i"] \end{tikzcd}$$

For NEC!: It is not hard to verify that $[\neg\Box p \vee \neg p](\neg\Box p \vee \neg p)$ is valid. From Proposition 14, $\Vdash ([p][\neg\Box p \vee \neg p](\neg\Box p \vee \neg p)) \leftrightarrow ([p \wedge (\neg\Box p \vee \neg p)](\neg\Box p \vee \neg p))$. From Proposition 15, $\Vdash ([p \wedge (\neg\Box p \vee \neg p)](\neg\Box p \vee \neg p)) \leftrightarrow ([p \wedge \neg\Box p](\neg\Box p \vee \neg p))$. However, $[p \wedge \neg\Box p](\neg\Box p \vee \neg p)$ is clearly not valid in the above (**S5**) model.

For RE! and RE: From the proof of the above case of NEC!, we have a valid equivalence: $([\neg\Box p \vee \neg p](\neg\Box p \vee \neg p)) \leftrightarrow \top$. However, although $[p]\top$ is still valid, $[p][\neg\Box p \vee \neg p](\neg\Box p \vee \neg p)$ is not valid, as we have shown. Therefore RE! is not valid w.r.t. $\Vdash$, thus RE is not valid either. □

From Lemma 19 and Theorem 18 we have:

Theorem 20 *None of* !COMP, NEC!, RE! *can be derived from* $\mathbf{PA} + \mathtt{PRE} + !\mathtt{K}' + !\mathtt{RE}$.

Proof From Theorem 18, for all ϕ: $\not\Vdash \phi$ implies $\nvdash_{\mathbf{PA}+\mathtt{PRE}+!\mathtt{K}'} \phi$. Moreover, since the rules in $\mathbf{PA} + \mathtt{PRE} + !\mathtt{K}'$ preserve validity, we can show that if a rule is not valid w.r.t. $\Vdash$, then it is not derivable in $\mathbf{PA} + \mathtt{PRE} + !\mathtt{K}'$. However, Lemma 19 says none of !COMP, NEC!, RE, RE! is valid w.r.t. $\Vdash$. □

Since DIST! is derivable from $\mathbf{PA} + \mathtt{PRE}$, the following corollaries are immediate:

Corollary 21 $\mathbf{PA} + \mathtt{DIST!} + !\mathtt{K}' + \mathtt{PRE} + !\mathtt{RE}$ *and its subsystems are not complete w.r.t.* $\vDash$.

Corollary 22 RE *is not an admissible rule of* **PA**.

Proof Theorem 11 says that **PA**+$\texttt{RE}$ is complete but Theorem 20 shows that **PA** cannot derive !$\texttt{COMP}$. □

Similar results hold for **PAK** w.r.t. **S5** frames.

Theorem 23 *None of* !$\texttt{COMP}$, $\texttt{NEC}$!, $\texttt{RE}$! *can be derived from* **PAK** + $\texttt{PRE}$ + !$\texttt{K}'$ + !$\texttt{RE}$ *thus* **PAK** + $\texttt{DIST}$! + !$\texttt{K}'$ + $\texttt{PRE}$ + !$\texttt{RE}$ *is not complete w.r.t.* $\vDash$ *on the class of* ***S5*** *frames.*

Proof We can easily check that $\texttt{T}$, $\texttt{4}$ and $\texttt{5}$ are valid under $\Vdash$. Moreover, Lemmata 17 and 16 still hold if restricted to **S5** models (we only need to pay attention to the inference rules). Finally Lemma 19 also holds in the **S5** setting since the counterexamples we mentioned are **S5**. □

To conclude this subsection, we give a *complete* axiomatization of PAL under $\Vdash$. Recall that !$\texttt{COMP}\wedge$ is the axiom schema $[\psi][\chi]\phi \leftrightarrow [\psi \wedge \chi]\phi$. We can show the completeness of **PA**+!$\texttt{COMP}\wedge$ w.r.t. our new semantics.

Theorem 24 **PA** + !$\texttt{COMP}\wedge$ *is sound and weakly complete w.r.t.* $\Vdash$.

Proof Soundness follows from Theorem 18 and Proposition 14. For completeness, clearly we can use the reduction axioms in **PA**+!$\texttt{COMP}\wedge$ to translate a PAL formula in to an equivalent EL formula w.r.t. $\Vdash$ (cf. the proof of Theorem 10). From Proposition 13 and the completeness of **K** w.r.t. $\vDash$, the desired completeness can be obtained. □

Remark 3 Despite the technical motivation behind **PA** + !$\texttt{COMP}\wedge$, it also stipulates a particular kind of update which may be reasonable in modelling real agents. What !$\texttt{COMP}\wedge$ says is that the agents are not "instant updaters" in the sense that they postpone the update until they hear all the consecutive announcements and collect them all together as a conjunction. Here are two realistic scenarios which may exemplify this rationale: 1. two announcements are made right after each other, and in the flash of time between the two, agents may not manage to update their information according to the first announcement. Therefore they may take the two announcements as a conjunction; 2. Agents may intentionally postpone the updates according to the announcements: it makes sense if we are considering announcements from different (reliable/unreliable) sources which may contradict each other. It is not hard to see that other postulates can also be implemented in the similar way through different context changing policies.

3.2 Another non-standard semantics

The rest of this section is devoted to the axioms $\texttt{DIST}$!, $\texttt{SDIST}$!, !$\texttt{K}'$, $\texttt{PRE}$ and the rule $\texttt{NEC}$!. First note that $\texttt{PRE}$ is valid w.r.t. the above semantics $\Vdash$. Thus $\texttt{DIST}$!, $\texttt{SDIST}$!, !$\texttt{K}'$ are also valid (by soundness of **PA** + $\texttt{PRE}$). We do not know yet whether these axioms are derivable from **PA**, and moreover it is unclear whether **PA** + $\texttt{NEC}$! is complete. To show that $\texttt{DIST}$!, $\texttt{SDIST}$!, !$\texttt{K}'$, and $\texttt{PRE}$ are not derivable in **PA** + $\texttt{NEC}$!, we now define another semantics ($\Vert\!\vdash$) which differs from $\vDash$ in the clause of $[\psi]\phi$.

In the sequel, we say that a formula ϕ is *special* if, modulo associativity and commutativity of $\wedge$, $\phi = \bigwedge_{1\le i\le n}\phi_i \wedge \bigwedge_{1\le j\le m}\phi'_j$ where $n \ge 1$, $m \ge 0$, and ϕ_i are in the shape of $[\chi]\chi'$ but none of ϕ'_j is in such a shape. If ϕ is special then we write $\phi = \phi_{[]} \wedge \phi_{-[]}$ where $\phi_{[]}$ and $\phi_{-[]}$ are the corresponding conjunctions of announcement formulas and non-announcement formulas respectively.

Given a Kripke model over **P**: $\mathcal{M} = (S, \to, V)$, the new truth condition for $[\psi]\phi$ is as follows:

$$\mathcal{M}, s \Vdash [\psi]\phi \;\Leftrightarrow\; \begin{cases} \mathcal{M}, s \Vdash \phi_{[]} & \text{if } \mathcal{M}, s \nVdash \psi \text{ and } \phi \text{ is special} \\ \mathcal{M}, s \Vdash \psi \text{ implies } \mathcal{M}|_{\psi}, s \Vdash \phi & \text{otherwise} \end{cases}$$

Intuitively, the new semantics for $[\psi]\phi$ depends on the exact form of ϕ thus $\mathtt{RE}!$ is expected to be invalid under this semantics. In the case that ψ is false and ϕ involves announcement formulas, we simply skip the absurd announcement of ψ (an agent does not go mad when hearing a false announcement followed by other announcements: they can just skip the first one).

According to this semantics, we can show that $\mathtt{DIST}!$, $\mathtt{SDIST}!$, $!\mathtt{K}'$ and $\mathtt{PRE}$ are not valid.

Lemma 25 $\mathtt{PRE}$*,* $\mathtt{DIST}!$*,* $\mathtt{SDIST}!$*, and* $!\mathtt{K}'$ *are not valid w.r.t.* $\Vdash$.

Proof For $\mathtt{PRE}$: Consider $(p \to [p][q]\neg q) \to [p][q]\neg q$ and the following (**S5**) model $\mathcal{M}$:

$$s : \neg p, q$$

It is clear that $\mathcal{M}, s \Vdash p \to [p][q]\neg q$. However, $\mathcal{M}, s \Vdash [p][q]\neg q \iff \mathcal{M}, s \Vdash [q]\neg q \iff (\mathcal{M}, s \Vdash q$ implies $\mathcal{M}|_q, s \Vdash \neg q)$. Thus $\mathcal{M}, s \nVdash [p][q]\neg q$.

For $\mathtt{DIST}!$ and $\mathtt{SDIST}!$: Consider the above model again, it is easy to verify that $[p](p \to [q]\neg q) \to ([p]p \to [p][q]\neg q)$ is not valid.

For $!\mathtt{K}'$: consider $[p]\Box[q]\bot \leftrightarrow (p \to \Box[p][q]\bot)$ and the following (**S5**) model:

$$s : p, \neg q \longleftrightarrow t : \neg p, q$$

$\mathcal{M}, s \Vdash [p]\Box[q]\bot \iff \mathcal{M}, s \Vdash p$ implies $(\mathcal{M}|_p, s \Vdash \Box[q]\bot)$, and $\mathcal{M}, s \Vdash p \to \Box[p][q]\bot \iff (\mathcal{M}, s \Vdash p$ implies $\mathcal{M}, s \Vdash \Box[p][q]\bot)$. Note that $\mathcal{M}, t \Vdash [p][q]\bot \iff \mathcal{M}, t \Vdash [q]\bot \iff \mathcal{M}|_q, t \Vdash \bot$. Therefore $\mathcal{M}, s \nVdash \Box[p][q]\bot$. Thus $\mathcal{M}, s \Vdash [p]\Box[q]\bot$ but $\mathcal{M}, s \nVdash p \to \Box[p][q]\bot$. □

Now we prove that **PA** is sound w.r.t. this semantics. Compared to $\vDash$, since we do not change the semantics for Boolean formulas and $\Box\phi$ formulas, the proof of Lemma 16 also works here w.r.t. $\Vdash$:

Lemma 26 $\mathtt{TAUT}$*,* $\mathtt{MP}$*,* $\mathtt{NECK}$ *and* $\mathtt{DISTK}$ *are valid w.r.t.* $\Vdash$.

Lemma 27 $!\mathtt{ATOM}$*,* $!\mathtt{NEG}$*,* $!\mathtt{CON}$*, and* $!\mathtt{K}$ *are valid w.r.t.* $\Vdash$.

Proof The case for $!\mathtt{ATOM}$ is trivial. $!\mathtt{CON}$ is a tricky one and we will see how the complicated case-divided semantics of $[\psi]\phi$ pays back.

For !CON: First note that $\phi \wedge \chi$ is not special iff ϕ and ψ are both not special. Now we consider two cases:

- If $\phi \wedge \chi$ is not special, then neither ϕ nor χ is special. $\mathcal{M}, s \Vdash [\psi](\phi \wedge \chi) \iff (\mathcal{M}, s \Vdash \psi$ implies $\mathcal{M}|_\psi, s \Vdash \phi \wedge \chi)$
$\iff \mathcal{M}, s \Vdash \psi$ implies $(\mathcal{M}|_\psi, s \Vdash \phi$ and $\mathcal{M}|_\psi, s \Vdash \chi)$
$\iff (\mathcal{M}, s \Vdash \psi$ implies $\mathcal{M}|_\psi, s \Vdash \phi)$ and $(\mathcal{M}, s \Vdash \psi$ implies $\mathcal{M}|_\psi, s \Vdash \chi)$
$\iff (\mathcal{M}, s \Vdash [\psi]\phi$ and $\mathcal{M}, s \Vdash [\psi]\chi)$
- If $\phi \wedge \chi$ is special then at least one of ϕ and χ is special. Suppose w.l.o.g. that χ is not special and ϕ is special thus $\phi = \phi_{[]} \wedge \phi_{-[]}$. Here are again two cases to be considered:
  - suppose $\mathcal{M}, s \Vdash \psi$ then the new semantics coincides with the standard one thus $\mathcal{M}, s \Vdash [\psi](\phi \wedge \chi) \leftrightarrow [\psi]\phi \wedge [\psi]\chi)$.
  - suppose $\mathcal{M}, s \nVdash \psi$ then $\mathcal{M}, s \Vdash [\psi](\phi \wedge \chi) \iff \mathcal{M}, s \Vdash \phi_{[]}$
  $\iff \mathcal{M}, s \Vdash \phi_{[]}$ and $(\mathcal{M}, s \Vdash \psi$ implies $\mathcal{M}|_\psi, s \Vdash \chi)$
  $\iff \mathcal{M}, s \Vdash [\psi]\phi$ and $\mathcal{M}, s \Vdash [\psi]\chi$

The proofs for !NEG and !K are almost as before under the standard semantics $\vDash$. We only need to handle the extra special cases. Now suppose ϕ is special. Clearly $\neg\phi$ and $\Box\phi$ are not special.

For !NEG: We only need to consider the case when $\mathcal{M}, s \nVdash \psi$ since otherwise the proof for the standard semantics suffices. Then it is clear that $\mathcal{M}, s \Vdash \psi \to \neg[\psi]\phi$ and $\mathcal{M}, s \Vdash [\psi]\neg\phi$ since $\neg\phi$ is not special. Thus $\mathcal{M}, s \Vdash [\psi]\neg\phi \leftrightarrow (\psi \to \neg[\psi]\phi)$.

For !K: It is clear that if $\mathcal{M}, s \nVdash \psi$ then $\mathcal{M}, s \Vdash [\psi]\Box\phi \leftrightarrow (\psi \to (\Box(\psi \to [\psi]\phi)))$. However, it does not suffice since even ψ is true at $\mathcal{M}, s$ it is still possible that $\psi \to (\Box(\psi \to [\psi]\phi))$ differs from the standard semantics due to the appearance of $[\psi]\phi$ in the scope of $\Box$. Now suppose $\mathcal{M}, s \Vdash \psi$. $\mathcal{M}, s \Vdash [\psi]\Box\phi \iff (\mathcal{M}, s \Vdash \psi$ implies $\mathcal{M}|_\psi, s \Vdash \Box\phi) \iff \mathcal{M}|_\psi, s \Vdash \Box\phi$. On the other hand, $\mathcal{M}, s \Vdash \psi \to \Box(\psi \to [\psi]\phi) \iff (\mathcal{M}, s \Vdash \psi$ implies $\mathcal{M}, s \Vdash \Box(\psi \to [\psi]\phi)) \iff \mathcal{M}, s \Vdash \Box(\psi \to [\psi]\phi) \iff (\forall t \rhd s : \mathcal{M}, t \Vdash \psi$ implies $(\mathcal{M}, t \Vdash [\psi]\phi)$. Note that the new semantics only differs from the standard one if ψ is false. Thus $(\forall t \rhd s : \mathcal{M}, t \Vdash \psi$ implies $(\mathcal{M}, t \Vdash [\psi]\phi) \iff (\forall t \rhd s : \mathcal{M}, t \Vdash \psi$ implies $(\mathcal{M}, t \Vdash \psi$ implies $\mathcal{M}|_\psi \Vdash \phi) \iff (\forall t \rhd s : \mathcal{M}, t \Vdash \psi$ implies $\mathcal{M}|_\psi, t \Vdash \phi) \iff (\forall t \rhd s : t$ exists in$\mathcal{M}|_\psi$implies $\mathcal{M}|_\psi, t \Vdash \phi) \iff \mathcal{M}|_\psi, s \Vdash \Box\phi$. □

Moreover, we can show that NEC! is valid w.r.t. $\Vdash$.

Lemma 28 NEC! *is valid under* $\Vdash$.

Proof Suppose $\Vdash \phi$. Now consider $[\psi]\phi$. There are two cases:

- ϕ is not special: Trivial.
- ϕ is special: It is in the shape of $\phi_{[]} \wedge \phi_{-[]}$. To verify $\mathcal{M}, s \Vdash [\psi]\phi$ there are again two cases. Suppose $\mathcal{M}, s \Vdash \psi$, then $\mathcal{M}, s \Vdash [\psi]\phi \iff \mathcal{M}|_\psi, s \Vdash \phi$ which is true since $\Vdash \phi$. Now suppose $\mathcal{M}, s \nVdash \psi$, then $\mathcal{M}, s \Vdash [\psi]\phi \iff \mathcal{M}, s \Vdash \phi_{[]}$. Since $\Vdash \phi$, $\Vdash \phi_{[]} \wedge \phi_{-[]}$ thus $\Vdash \phi_{[]}$. Therefore, $\mathcal{M}, s \Vdash \phi_{[]}$. This concludes the proof. □

Lemmata 26, 27, and 28 showed that **PA** + NEC! is sound w.r.t. $\Vdash$. Together with 25 we have:

Theorem 29 *None of* DIST!, SDIST! !K′, PRE *can be derived from* **PA** + NEC!.

As an immediate corollary:

Corollary 30 **PA** + NEC! *is not complete w.r.t. standard semantics* $\vDash$.

Similar to Theorem 23, it is not hard to verify that the above results also hold for the **S5** setting:

Theorem 31 *None of* DIST!, SDIST! !K′, PRE *can be derived from* **PAK** + NEC! *thus* **PAK** + NEC! *is not complete w.r.t. standard semantics* $\vDash$ *on the class of* **S5** *frames.*

Before moving to the second theme of this paper, we would like to summarize the results so far as follows (**PA** can be replaced by **PAK**):

Derivable/admissible in **PA**	Not derivable/admissible in **PA**
WDIST!, PFUNC, RE$\neg$, RE$\wedge$, RE$\Box$	!COMP, DIST!, SDIST!, PRE, !K′, NEC!, RE!, RE
Sound & Complete systems	Sound & Incomplete systems
PA-!CON+DIST!+NEC!, **PA**+PRE+NEC! **PA**+RE, **PA**+!COMP	**PA**+!K′+PRE+DIST!+!RE, **PA**+NEC!

4 An alternative axiomatization of PAL without reductions

In this section, we propose an alternative axiomatization **PAN** without using the previously mentioned reduction axioms. The completeness is proved directly by using canonical model similar to the method used in Gerbrandy and Groeneveld (1997). The key idea is to treat $[\psi]$ as a usual modality of normal modal logics and use axioms to characterize the *update transitions* among epistemic models. Similar ideas of viewing updates as transitions among models in a "super model" also appeared, under different contexts, in Baltag and Moss (2004), van Benthem (2007), van Benthem et al. (2009), and Holliday et al. (2012). We will come back to some of these works in Sect. 5.

4.1 An auxiliary semantics for PAL

This subsection explores the idea of treating $[\psi]$ as a usual modality interpreted on models with ψ transitions. Let us begin with an extension of the standard Kripke model.

Definition 32 (Extended model) An *extended (Kripke) model* $\mathcal{M}$ for PAL is a tuple $(S, \to, \{\xrightarrow{\psi} \mid \psi \in \text{PAL}, V)$ where:

– $(S, \to, V)$ is a standard Kripke model for PAL.

– For each ψ, $\overset{\psi}{\to}$ is a (possibly empty) binary relation over S.

We call $(S, \to, V)$ the *Kripke core* of $\mathcal{M}$ (notation $\mathcal{M}^-$).

We now define an auxiliary semantics for PAL on extended models $(S, \to, \{\overset{\psi}{\to} \mid \psi \in \text{PAL}, V)$:

$$\begin{array}{rcl}
\mathcal{M}, s \vDash^a \top & \Leftrightarrow & \text{always} \\
\mathcal{M}, s \vDash^a p & \Leftrightarrow & s \in V(p) \\
\mathcal{M}, s \vDash^a \neg\phi & \Leftrightarrow & \mathcal{M}, s \nvDash^a \phi \\
\mathcal{M}, s \vDash^a \phi \wedge \psi & \Leftrightarrow & \mathcal{M}, s \vDash^a \phi \text{ and } \mathcal{M}, s \vDash^a \psi \\
\mathcal{M}, s \vDash^a \Box\phi & \Leftrightarrow & \forall t : s \to t \text{ implies } \mathcal{M}, t \vDash^a \phi \\
\mathcal{M}, s \vDash^a [\psi]\phi & \Leftrightarrow & \forall t : s \overset{\psi}{\to} t \text{ implies } \mathcal{M}, t \vDash^a \phi
\end{array}$$

Note that $[\psi]$ is interpreted similarly to the $\Box$ modality with usual semantics. We may interpret PAL on extended models under the standard semantics by setting $\mathcal{M}, s \vDash \phi \iff \mathcal{M}^-, s \vDash \phi$ for any pointed extended model $\mathcal{M}, s$ and any PAL formula ϕ. It is clear that, in general, the auxiliary semantics does not coincide with the standard semantics on announcement formulas, e.g., consider the following pointed extended model $\mathcal{M}, s$: $\mathcal{M}, s \vDash^a \langle\neg p\rangle\top$, but $\mathcal{M}^-, s \nvDash \langle\neg p\rangle\top$ thus $\mathcal{M}, s \nvDash \langle\neg p\rangle\top$.

$$\mathcal{M}: \quad s : p \xrightarrow{\neg p} s' : \neg p \qquad\qquad \mathcal{M}^-: \quad s : p \qquad\qquad s' : \neg p$$

In the following, we consider a class of extended models where the two semantics coincide.

Definition 33 (*Normal extended Kripke model*) An extended model $\mathcal{M} = (S, \to, \{\overset{\psi}{\to} \mid \psi \in \text{PAL}, V)$ for PAL is called *normal* if the following properties hold for any s, t in $\mathcal{M}$:

U-Functionality For any PAL formula ψ: If $\mathcal{M}, s \vDash^a \psi$, then s has a unique ψ-successor. If $\mathcal{M}, s \nvDash^a \psi$ then s has no outgoing ψ-transition.

U-Invariance if $s \overset{\psi}{\to} t$ then for all $p \in \mathbf{P} : s \in V(p) \iff t \in V(p)$.

U-Zig if $s \to s'$, $s \overset{\psi}{\to} t$ and $s' \overset{\psi}{\to} t'$ then $t \to t'$.

U-Zag if $t \to t'$ and $s \overset{\psi}{\to} t$ then there exists an s' such that $s \to s'$ and $s' \overset{\psi}{\to} t'$.

An extended model is called ψ*-normal* if it enjoys the last 3 properties and has the functionality property for a particular ψ (ψ**-Functionality**).

In the above definition **U** stands for "update". ψ**-Functionality** says that the ψ-update is a partial function depends on whether ψ can be executed or not. **U-Invariance** says that the update should not change the valuation of the states. The last two properties together define the updated relations and are best illustrated by the following diagrams of commutativity:

$$\begin{tikzcd}
s \arrow[r] \arrow[d, "\psi"] & s' \arrow[d, "\psi"] & \textbf{U-Zig} \arrow[d, phantom, "\Longrightarrow"] & s \arrow[r] \arrow[d, "\psi"] & s' \arrow[d, "\psi"] & \textbf{U-Zag} \arrow[d, phantom, "\Longleftarrow"] & s \arrow[d, "\psi"] & \\
t & t' & & t \arrow[r] & t' & & t \arrow[r] & t'
\end{tikzcd}$$

In the setting of temporal epistemic logics such as Fagin et al. (1995), the properties like **U-Zag** are often called *(synchronous) perfect recall* or *no forgetting* (cf. Halpern et al. 2004). On the other hand, the properties like **U-Zig** are often called *no miracles* in the DEL literature (cf. e.g., van Benthem et al. 2009). We will come back to these two properties in Sect. 5. We give the properties the names **U-Zig** and **U-Zag** here because they play an important role in proving a bisimulation lemma, which involves the checking of **Zig** and **Zag** conditions in the following definition of bisimulation.

Definition 34 (*Bisimulation*) A binary relation Z is called a *bisimulation* between two pointed Kripke models $\mathcal{M}, s$ and $\mathcal{N}, t$, if sZt and whenever wZv the following hold:

Invariance $p \in V^{\mathcal{M}}(w)$ iff $p \in V^{\mathcal{N}}(v)$,
Zig if $w \to w'$ for some w' in $\mathcal{M}$ then there is a $v' \in S^{\mathcal{N}}$ with $v \to v'$ and $w'Zv'$,
Zag if $v \to v'$ for some v' in $\mathcal{N}$ then there is a $w' \in S^{\mathcal{M}}$ with $w \to w'$ and $w'Zv'$.

It is a standard result that the PAL formulas are preserved under bisimulation (cf. e.g., van Ditmarsch et al. 2007).

Lemma 35 *Given a* PAL *formula ψ and a ψ-normal extended model $\mathcal{M}$, we have:*

$$\mathcal{M}^-|_\psi, w \leftrightarroweq \mathcal{M}^-, v$$

if the following two conditions hold:

1. $w \xrightarrow{\psi} v$ *in* $\mathcal{M}$,
2. *for every point u in $\mathcal{M}$, $\mathcal{M}^-, u \vDash \psi \iff \mathcal{M}, u \vDash^a \psi$.*

Proof Let Z be the binary relation between $\mathcal{M}^-|_\psi$ and $\mathcal{M}^-$ such that sZt iff $s \xrightarrow{\psi} t$ in $\mathcal{M}$. Clearly, Z is non-empty since $w \xrightarrow{\psi} v$ in $\mathcal{M}$. Now suppose sZt (thus $s \xrightarrow{\psi} t$ in $\mathcal{M}$), we need to check the three conditions of bisimulation. The invariance condition is guaranteed by **U-Invariance**. For **Zig**, suppose $s \to s'$ in $\mathcal{M}^-|_\psi$ then it is clear that $s \to s'$ in $\mathcal{M}^-$ and $\mathcal{M}^-, s' \vDash \psi$. According to the second assumption, $\mathcal{M}, s' \vDash^a \psi$. Thus from **$\psi$-Functionality**, there is t' such that $s' \xrightarrow{\psi} t'$. And from **U-Zig** we have $t \to t'$ in $\mathcal{M}^-$. Now for **Zag**, suppose $t \to t'$ for some t' in $\mathcal{M}^-$. From **U-Zag**, there is an s' in $\mathcal{M}$ such that $s \to s'$ and $s' \xrightarrow{\psi} t'$. By **ψ-Functionality** $\mathcal{M}, s' \vDash^a \psi$. According to the second assumption again, $\mathcal{M}^-, s' \vDash \psi$ thus $s \to s'$ exists in $\mathcal{M}^-|_\psi$. $\square$

In words, this lemma says that in a normal extended Kripke model with some conditions, *updating* with a ψ-announcement has the same effect as *moving* along the ψ-transition. Roughly speaking, we can turn the *dynamics (model transformations)* into *statics (state transitions)* in the normal extended model.

Now we can establish the equivalence of the auxiliary semantics and the standard semantics on normal extended models.

Theorem 36 *For any* PAL *formula ϕ and any normal extended Kripke model $\mathcal{M}$:*

$$\mathcal{M}, s \vDash^a \phi \iff \mathcal{M}^-, s \vDash \phi$$

Proof We prove it by induction on the structure of the formulas. Since $\vDash^a$ on extended models coincides with $\vDash$ for announcement-free formulas, the cases for Boolean combinations and $\Box\phi$ are trivial.

For the case of $[\psi]\phi$, we distinguish two sub-cases depending on the truth value of ψ. Suppose $\mathcal{M}, s \nvDash^a \psi$ then by the induction hypothesis (IH) $\mathcal{M}^-, s \nvDash \psi$ thus according to the standard semantics of PAL, $\mathcal{M}^-, s \vDash [\psi]\phi$. Since $\mathcal{M}$ is a normal extended Kripke model and $\mathcal{M}, s \nvDash^a \psi$, by **U-Functionality** there is no outgoing ψ-transition from s in $\mathcal{M}$, therefore $\mathcal{M}, s \vDash^a [\psi]\phi$.

Now we consider the case of $\mathcal{M}, s \vDash^a \psi$. By **U-Functionality**, there must be a unique ψ-successor of s in $\mathcal{M}$ (call it t). From IH and Lemma 35, $\mathcal{M}^-|_\psi, s \leftrightarroweq \mathcal{M}^-, t$. Note that it is a bisimilarity between standard Kripke models without ψ-transitions. Since PAL formulas are invariant under bisimulation, thus for any PAL formula ϕ we have $\mathcal{M}^-|_\psi, s \vDash \phi \iff \mathcal{M}^-, t \vDash \phi$. From IH, $\mathcal{M}^-|_\psi, s \vDash \phi \iff \mathcal{M}, t \vDash^a \phi$. According to the semantics and **U-Functionality**, it is clear that

$$\mathcal{M}, s \vDash^a [\psi]\phi \iff \mathcal{M}^-, s \vDash [\psi]\phi.$$

□

4.2 System **PAN** and its completeness

Let us recall the important axiom schemata in **PAN** besides the "usual suspects", `DIST!` and `NEC!`:

`INV`	$(p \to [\psi]p) \wedge (\neg p \to [\psi]\neg p)$
`PFUNC`	$\langle\psi\rangle\phi \leftrightarrow (\psi \wedge [\psi]\phi)$
`NM`	$\Diamond\langle\psi\rangle\phi \to [\psi]\Diamond\phi$
`PR`	$\langle\psi\rangle\Diamond\phi \to \Diamond\langle\psi\rangle\phi$

Intuitively, axioms `INV`, `PFUNC`, `NM` and `PR` correspond to the properties of normal extended models and altogether they "define" the updated model after an announcement. We will get back to the meaning of these axioms in Sect. 5. Note that since **PAN** includes `NEC!` and `DIST!`, `RE` is admissible in **PAN** (cf. the Proof of Proposition 8).

We first prove a simple theorem in **PAN** to be used later.

Proposition 37 $\neg\psi \to [\psi]\phi$ *is derivable in* **PAN**.

Proof

$$\begin{array}{lll}
1 & \vdash_{\mathbf{PAN}} \langle\psi\rangle\neg\phi \leftrightarrow (\psi \wedge [\psi]\neg\phi) & \texttt{PFUNC} \\
2 & \vdash_{\mathbf{PAN}} [\psi]\neg\neg\phi \leftrightarrow (\psi \rightarrow \langle\psi\rangle\phi) & \texttt{TAUT} \\
3 & \vdash_{\mathbf{PAN}} [\psi]\phi \leftrightarrow (\psi \rightarrow \langle\psi\rangle\phi) & \texttt{RE} \\
4 & \vdash_{\mathbf{PAN}} \neg\psi \rightarrow (\psi \rightarrow \langle\psi\rangle\phi) & \texttt{TAUT} \\
5 & \vdash_{\mathbf{PAN}} \neg\psi \rightarrow [\psi]\phi & \texttt{MP}(3,4)
\end{array}$$

□

It is easy to show the soundness of **PAN**.

Proposition 38 *The system* **PAN** *is sound w.r.t. the standard semantics.*

We may prove the (strong) completeness by showing that all the axioms of the complete system **PA** + `DIST!` + `NEC!` can be proved in **PAN**. However, in the rest of this section we will prove the completeness directly without referring to any other completeness result.

A canonical model based proof to the completeness of a normal logic w.r.t. a class $\mathbb{C}$ of structures usually consists of the following steps:

1. Prove the *Lindenbaum*-like lemma: every consistent set of formulas can be extended into a maximal consistent set (MCS).
2. Construct the canonical model.
3. Prove the *truth lemma*: a formula is true at a state in the canonical model iff it is in the state (an MCS).
4. Show that the canonical model is indeed based on some structure in $\mathbb{C}$.

As for PAL, to prove the Lindenbaum-like lemma w.r.t. **PAN** is a routine task. We can also define the same canonical Kripke model as the tuple $(S^c, \rightarrow^c, V^c)$ where:

- S^c is the set of all the maximal consistent set w.r.t. **PAN**
- $s \rightarrow^c t$ iff (for all ϕ: $\phi \in t$ implies $\Diamond\phi \in s$)
- $V^c(p) = \{s \mid p \in s\}$

The main difficulty comes in proving the truth lemma. For any formula without announcement operators, by induction on the structure of formulas, for any MCS s in $\mathcal{M}^c$ we can show that (cf. e.g., Blackburn et al. 2002):

$$\mathcal{M}^c, s \vDash \phi \iff \phi \in s$$

Now, how to prove the case for announcement formulas $[\psi]\phi$?

Our strategy is to make a *detour* by using the auxiliary semantics. More precisely, the proof proceeds as follows:

Proof Strategy The detour consists of:

1. Construct an extended canonical model with update transitions.
2. Show that the truth lemma holds under auxiliary semantics.
3. Establish the equivalence between the standard semantics and the auxiliary semantics on the extended canonical model by using axioms that define the updates.

4. Finally we obtain the truth lemma w.r.t. the standard semantics and standard canonical model and completeness follows easily.

Let us begin with the extension of the standard canonical model.

Definition 39 (Extended canonical model). The extended canonical model $\mathcal{M}^c_+$ for **PAN** is a tuple $(S^c, \to^c, \{\xrightarrow{\psi} \mid \psi \in \text{PAL}, V^c)$ where:

- $(S^c, \to^c, V^c)$ is a standard canonical model for **PAN**,
- $s \xrightarrow{\psi} t$ iff for all ϕ: $\phi \in t$ implies $\langle\psi\rangle\phi \in s$.

It is easy to see that the Kripke core of $\mathcal{M}^c_+$ is just the standard canonical model, namely $(\mathcal{M}^c_+)^- = \mathcal{M}^c$.

Based on the above definition, it is straightforward to show the following:

Proposition 40 *$s \xrightarrow{\psi} t$ iff for all $\phi : [\psi]\phi \in s$ implies $\phi \in t$.*

Since **PAN** includes `NEC!` and `DIST!` the following truth lemma is just a standard exercise for normal modal logic (cf. e.g., Blackburn et al. 2002):

Lemma 41 (Truth lemma w.r.t. $\vDash^a$). *For any* PAL *formula ϕ:*

$$\mathcal{M}^c_+, s \vDash^a \phi \iff \phi \in s$$

In the following we study the properties of $\mathcal{M}^c_+$.

Proposition 42 *For any s in $\mathcal{M}^c_+$, $s \xrightarrow{\psi} t$ then for all $p \in \mathbf{P}$: $p \in s \iff p \in t$.*

Proof Straightforward from the axiom `INV` and Proposition 40. □

Proposition 43 *For any s in $\mathcal{M}^c_+$, s has at most one ψ-successor.*

Proof Suppose s has two different ψ-successors t and t'. Since t and t' are two different MCSs, then there exists a ϕ such that $\phi \in t$ and $\neg\phi \in t'$, then according to the definition of $\xrightarrow{\psi}$ we have $\langle\psi\rangle\phi \in s$. By axiom `PFUNC` and `MP`, we have $[\psi]\phi \in s$. From Proposition 40 and the fact that t' is a ψ-successor of s, $\phi \in t'$, contradictory to the assumption that $\neg\phi \in t'$ and t' is an MCS. □

Proposition 44 *For any* PAL *formula ψ: if $\psi \in s$ then s must have a unique ψ-successor t and $t = \{\phi \mid \langle\psi\rangle\phi \in s\} = \{\phi \mid [\psi]\phi \in s\}$. If $\psi \notin s$ then s does not have any ψ-successor.*

Proof Note that $\top \in s$, and from `NEC!` we have $[\psi]\top \in s$ for any ψ. Now if $\psi \in s$ then from axiom `PFUNC` we have $\langle\psi\rangle\top \in s$. We claim that $t = \{\phi \mid [\psi]\phi \in s\}$ is a maximal consistent set. First we prove it is consistent w.r.t. **PAN**. Suppose not, then there are $\phi_1 \ldots \phi_n \in t$ such that $\vdash_{\mathbf{PAN}} \phi_1 \wedge \cdots \wedge \phi_n \to \bot$. From `NEC!` and `DIST!`, it follows that $\vdash_{\mathbf{PAN}} ([\psi]\phi_1 \wedge \cdots \wedge [\psi]\phi_n) \to [\psi]\bot$. Thus $[\psi]\bot \in s$ which is contradictory to $\langle\psi\rangle\top \in s$ and the fact that s is a consistent set. Next we prove t is maximal. Suppose not, then there exists a PAL formula ϕ such that $\phi \notin t$ and $\neg\phi \notin t$, therefore neither $[\psi]\phi$ nor $[\psi]\neg\phi$ is in s. Since s is maximal, $\neg[\psi]\neg\phi \in s$ namely

$\langle\psi\rangle\phi \in s$. From axiom $\mathtt{PFUNC}$, $[\psi]\phi$ is in s, contradiction. In sum, $\{\phi \mid [\psi]\phi \in s\}$ is a maximal consistent set. According to the definition of $\xrightarrow{\psi}$, it is clear that $s \xrightarrow{\psi} t$. Since $\psi \in s$ it is easy to see that $\{\phi \mid \langle\psi\rangle\phi \in s\} = \{\phi \mid [\psi]\phi \in s\}$. From Proposition 43, t is the unique ψ-successor of s.

For the second claim, suppose $\psi \notin s$ and $s \xrightarrow{\psi} t$ then $\langle\psi\rangle\top \in s$ for $\top \in t$. However, from the fact that $\psi \notin s$ we have $\neg\psi \in s$ thus by Proposition 37 we have $\neg\langle\psi\rangle\top \in s$, contradiction. □

Due to Lemma 41 and the above proposition we have:

Proposition 45 *$\mathcal{M}^c_+$ has the* ***U-Functionality****.*

In the following we show that $\mathcal{M}^c_+$ has the properties of **U-Zig** and **U-Zag**.

Proposition 46 *In $\mathcal{M}^c_+$, if $s \to s'$, $s \xrightarrow{\psi} t$ and $s' \xrightarrow{\psi} t'$ then $t \to t'$.*

Proof In order to show $t \to t'$, we take an arbitrary $\phi \in t'$ and show that $\Diamond\phi \in t$. Suppose $\phi \in t'$, then $\langle\psi\rangle\phi \in s'$ for $s' \xrightarrow{\psi} t'$. Thus $\Diamond\langle\psi\rangle\phi \in s$ for $s \to s'$. Due to axiom $\mathtt{NM}$, we have $[\psi]\Diamond\phi \in s$. Since $s \xrightarrow{\psi} t$, $\Diamond\phi \in t$. □

Proposition 47 *In $\mathcal{M}^c_+$, if $t \to t'$ and $s \xrightarrow{\psi} t$ then there exists an s' such that $s \to s'$ and $s' \xrightarrow{\psi} t'$.*

Proof Let $X = \{\langle\psi\rangle\phi \mid \phi \in t'\} \cup \{\phi \mid \Box\phi \in s\}$. It is clear that if X is consistent then it can be extended into a desired maximal consistent set. We just need to show that X is consistent. Suppose not, there are $\phi_0 \ldots \phi_n \in t'$ and $\Box\chi_0, \ldots, \Box\chi_m \in s$ such that $\vdash_{\mathbf{PAN}} (\langle\psi\rangle\phi_0 \wedge \cdots \wedge \langle\psi\rangle\phi_n \wedge \chi_0 \wedge \cdots \wedge \chi_m) \to \bot$. By tautologies, $\vdash_{\mathbf{PAN}} (\chi_0 \wedge \cdots \wedge \chi_m) \to \neg(\langle\psi\rangle\phi_0 \wedge \cdots \wedge \langle\psi\rangle\phi_n)$ thus $\vdash_{\mathbf{PAN}} (\chi_0 \wedge \cdots \wedge \chi_m) \to ([\psi]\neg\phi_0 \vee \cdots \vee [\psi]\neg\phi_n)$.

From $\mathtt{NECK}$ and $\mathtt{DISTK}$, we have:

$$\vdash_{\mathbf{PAN}} \Box(\chi_0 \wedge \cdots \wedge \chi_m) \to \Box([\psi]\neg\phi_0 \vee \cdots \vee [\psi]\neg\phi_n)$$

Since $\Box\chi_0, \ldots, \Box\chi_m \in s$, $\Box([\psi]\neg\phi_0 \vee \cdots \vee [\psi]\neg\phi_n) \in s$. By $\mathtt{DIST!}$, $\mathtt{NEC!}$, $\mathtt{DISTK}$ and $\mathtt{NECK}$, it is easy to show that $\Box[\psi](\neg\phi_0 \vee \cdots \vee \neg\phi_n) \in s$. The contrapositive of $\mathtt{PR}$ is $\Box[\psi]\phi \to [\psi]\Box\phi$, thus $[\psi]\Box(\neg\phi_0 \vee \cdots \vee \neg\phi_n) \in s$. Since $s \xrightarrow{\psi} t$, $\Box(\neg\phi_0 \vee \cdots \vee \neg\phi_n) \in t$. By the fact that $t \to t'$, $\neg\phi_0 \vee \cdots \vee \neg\phi_n \in t'$, contradictory to $\phi_0, \ldots, \phi_n \in t'$. □

In sum, $\mathcal{M}^c_+$ has all the properties of a normal extended model:

Lemma 48 *$\mathcal{M}^c_+$ is a normal extended Kripke model.*

From Lemma 48 and Theorem 36, we have for any PAL formula ϕ, any $s \in \mathcal{M}^c_+$:

$$\mathcal{M}^c_+, s \vDash^a \phi \iff (\mathcal{M}^c_+)^-, s \vDash \phi$$

Now based on Lemma 41 and the fact that $(\mathcal{M}^c_+)^- = \mathcal{M}^c$, we can safely make the following conclusion:

Lemma 49 *For any* PAL *formula* ϕ *and a point* s *in* $\mathcal{M}^c$*:*

$$\phi \in s \iff \mathcal{M}^c, s \vDash \phi$$

Based on Lemma 49, every **PAN** consistent set of formulas has a model. Thus the strong completeness follows.

Theorem 50 **PAN** *is sound and strongly complete w.r.t. the standard semantics of* PAL *on the class of all Kripke frames.*

Since **S5** models are closed under updates of public announcement (cf e.g. van Ditmarsch et al. 2007), and **S5** axioms are canonical, we can prove the following completeness theorems for PAL on various class of sub-**S5** frames as easy corollaries of Theorem 50.

Corollary 51 **PAN**+T*,* **PAN**+T+4 *and* **PAN**+T+4+5 *are sound and strongly complete w.r.t. the standard semantics of* PAL *on the class of all* ***T****,* ***S4****,* ***S5*** *frames.*

5 Discussion

Technically, INV, PFUNC, NM and PR are used to force the extended canonical model to be normal. The corresponding properties to these axioms are used to prove a crucial bisimulation lemma (Lemma 35): properties for INV, NM, PR correspond to the three conditions of bisimulation while PFUNC says the updates are partial-functional. It is not hard to see that these four together explicitly states that the public announcement updates are *partial p-morphisms* over the space of all Kripke models. This correspondence result was first shown by van Benthem (2012) in the setting of reduction axioms based on earlier insights in (van Benthem 2007).[10] In this section, instead of taking a holistic view of theses axioms, we would like to divide these four axioms into two groups and discuss them separately. Before going into details, we first summarize our points as follows:

- INV, PFUNC and their variations are crucial for the reduction approach in dynamic epistemic logics, but at the same time they also introduce some technical drawbacks and limitations of the modelling power. In a more general setting, we may leave out these restrictions and still obtain complete axiomatizations.
- NM and PR are about the epistemic effects of the announcements (or, say, the ability of the agents). According to our point of view, they are more essential in dynamic epistemic logics. PR is a well-known assumption for agents and NM amounts to a special property shared by many dynamic epistemic logics, which may distinguish them from the usual epistemic temporal logics.

[10] van Benthem (2007) views the reduction axioms as postulates of abstract updates thus opens a new kind of correspondence study in modal logic: between axioms and updates.

5.1 INV and PFUNC: propositional invariance and partial-functionality

Bearing **PAN** in mind, we may have new readings of the reduction axioms: !NEG is actually equivalent to PFUNC if you consider its contrapositive and thus denoting the partial-functionality of the updates; !ATOM expresses the invariance of basic propositions under the presence of PFUNC or !NEG; !CON is not essential since it follows easily form NEC! and DIST!; finally !K (or !K$'$) combines NM and PR together under the presence of PFUNC or !NEG, as observed in van Benthem (2011, Chap. 11) and van der Hoek and Pauly (2006).

To make the reduction work, the invariance of basic propositions and partial-functionality are necessary in order to handle the basic case for p and to swap the announcement operator and negation. In other dynamic epistemic logics which admit reductions, e.g., the event model approach as in Baltag et al. (1998), these two properties also show up in one way or another.[11]

However, are these two properties really essential in dealing with general logical dynamics? Or, shall we restrict ourselves to the dynamics having these two properties? The answers may be "*No*", if we take a more liberal view departing from the method based on reductions. In the rest of this subsection, we will look at these two properties in a more technical point of view.

First of all, properties like propositional invariance are responsible to the loss of uniform substitution in dynamic epistemic logics. It depends on your view to argue whether it is a downside for a logic, but it may not be a desired must-have for a new logic. Actually, it had been a major open problem to axiomatize the valid PAL formulas that are closed under uniform substitution. Holliday et al. (2012) gave a complete axiomatization of this "substitution core" assuming infinitely many agents. Their axiomatization is very similar to our **PAN** in spirit but without INV. However, just deleting INV from **PAN** is not enough for the completeness. Two compensations should be added: reflexivity for $\langle\top\rangle$: $p \rightarrow \langle\top\rangle p$ and the composition axiom !COMP. The addition of the first one is easy to understand due to the absence of INV. For the second one, note that !COMP is derivable in **PAN** due to its completeness, but this has to be done via a reduction-and-assembling process just like in **PA** + RE or **PA** + DIST! + NEC!. Now with the missing INV, the full reduction is not possible any more thus !COMP is called for to represent a special kind of transitivity of the update modalities. To accommodate the substitution core of PAL and to prove the completeness, Holliday et al. (2012) independently proposes a general semantics on a class of *legal models* which are similar to our normal extended models (but with different conditions).

Now let us look at the partial-functionality and the corresponding axiom schema PFUNC. To be more precise, we may split PFUNC into *two* parts: $\langle\psi\rangle\phi \rightarrow [\psi]\phi$ and

[11] In some dynamic epistemic logics, such as van Benthem et al. (2006), the valuation of basic propositions can also be updated in a systematic way based on the previous valuation. However, this does not change the picture dramatically. Such logics with factual changes also validate axioms similar to INV thus still suffering from the loss of uniform substitution.

$\langle\psi\rangle\top \leftrightarrow \psi$ representing respectively the functionality and the precondition of the update.[12]

$\langle\psi\rangle\phi \rightarrow [\psi]\phi$ says that there is just one *deterministic* outcome of a truthful announcement. It is reasonable in the setting of public announcement, but may not be useful in the more general setting. Executing an action may have *non-deterministic* effects due to some external factors that are not modelled in the framework. In computer science, non-determinism is one of the most important issues, and it is useful to work with non-deterministic programs even though the actual programs are often deterministic. By using non-deterministic programs, we can talk about the consequences of actions more easily, e.g., we may want to know whether some finite iteration of an communicative action may let us know something eventually.[13] However, the combination of deterministic atomic actions and arbitrary finite iteration may have serious computational costs under certain conditions.[14] For example, Harel (1985) showed that the satisfiability problem of PDL with intersection is doubly-exponential while *deterministic* PDL with intersection is highly undecidable (Π^1_1-hard). Similar things happen in the setting of PAL with iterations. Miller and Moss (2005) showed that PAL with iteration is also highly undecidable via tiling arguments, even when restricted to some very simple fragments. The deterministic action helps in coding the grid which is crucial for an unbounded tiling argument of the undecidability.

On other hand, $\langle\psi\rangle\top \leftrightarrow \psi$ specifies the preconditions of the announcements, i.e., the sufficient and necessary condition for an announcement to be executable. Similar preconditions also play important roles in the event model approach of dynamic epistemic logic as in Baltag et al. (1998). However, there are also cases where we cannot specify the exact preconditions of the actions in terms of a simpler formula. For example, if we consider PAL with protocols as in van Benthem et al. (2009), being truthful is just a *necessary* condition for an announcement to be executable, since the announcement should also comply with the protocol constraints.

Now, what if we drop the properties of propositional invariance and the partial-functionality in modelling certain logical dynamics? For axiomatization, it seems that the reduction method cannot work any more, but what about our detour method w.r.t. non-reduction axioms? In the PAL setting, both properties are used to obtain a bisimulation lemma. However, according to our general proof strategy on Sect. 4.2, we just need to show the standard semantics and the auxiliary semantics do coincide on canonical models, while the bisimulation argument is not obligatory. A good example is the *Epistemic Action Logic* developed by Wang and Li (2012) where the actions can be non-deterministic and the truth values of basic propositions are not preserved or computed after executing the actions. In spirit, it is still a dynamic epistemic logic since the interpretation of the action modalities are *model transformers*. This approach also differs from the usual DEL logics in the sense that the updated model is computed not only from the previous epistemic information but also from the temporal information in the model. An axiomatization including the axioms of perfect recall and a version

[12] We conjecture that weakening `PFUNC` to $\langle\psi\rangle\top \leftrightarrow \psi$ is enough to make the system complete. The functionality property can be guaranteed under the presence of `INV`, `NM`, and `PR`.

[13] Recall the famous muddy children example, cf. e.g., van Ditmarsch et al. (2007).

[14] See Goldblatt and Jackson (2012) for a more detailed discussion on the reason for the undecidability.

of no miracles is provided, and the completeness is proved following the general proof strategy sketched in this paper. A notable feature of the axiomatization is that it admits *uniform substitution*, in contrast to the usual dynamics epistemic logics. We conjecture that if we extend the language in Wang and Li (2012) with iteration of actions, the logic is still decidable, contrasting the undecidability of the iterated public announcement logic as in Miller and Moss (2005).

5.2 NM and PR: no miracles and perfect recall

PAN is similar to the axiomatization given in Gerbrandy and Groeneveld (1997, System **CK**). Despite the differences caused by the apparently quite different semantics and the use of the auxiliary semantics in our proof,[15] Gerbrandy and Groeneveld (1997) used the "Generalized Ramsey Axiom" (similar to !K) to capture the crucial interaction between announcements and knowledge. In contrast, our NM and PR axioms explicitly express the commutativity properties that are needed for establishing the equivalence of the standard semantics and the auxiliary semantics on canonical models.

In fact, there are deeper reasons to advocate NM and PR axioms rather than !K. First of all, NM and PR are no strangers to (temporal) epistemic logicians. In the filed of *Epistemic Temporal Logic* (ETL) initiated by Fagin et al. (1995) and Parikh and Ramanujam (1985), PR corresponds to the property of *synchronous perfect recall* if we take $\langle\psi\rangle$ as a one step action operator $\langle a\rangle$. On the other hands, NM is very similar to the following axiom:[16]

$$\mathtt{NL} : \Diamond\langle a\rangle\phi \to \langle a\rangle\Diamond\phi$$

which corresponds to the property of *synchronous no learning* (cf. e.g., Halpern et al. 2004 and the references therein).[17]

Synchronous no learning and perfect recall roughly characterize the agents who satisfy the following two postulates respectively:[18]

- if they know ϕ after an action then it must be the case that they already expected ϕ before the action.
- if they expect to learn ϕ after an action then they can indeed learn ϕ after the execution of the action.

[15] The semantics for PAL as in Gerbrandy and Groeneveld (1997) is based on "possibilities" which are essentially bisimulation classes of pointed Kripke models. The public/private announcement operator is defined as a function mapping one possibility to another by essentially deleting epistemic relations, thus every announcement is executable. Essentially, the formulas are interpreted in a "universal" model where each point stands for a class of Kripke models.

[16] In the temporal epistemic setting based on linear temporal logic, the axiom is usually presented as $\Diamond \bigcirc \phi \to \bigcirc\Diamond\phi$ where $\bigcirc$ is the next moment operator. In such a setting, there is no difference between the "box" and "diamond" forms of the $\bigcirc$ operator since it is assumed that there is always a unique next moment.

[17] Halpern et al. (2004) discussed the general properties of perfect recall and no learning, which can be simplified significantly in the setting of synchronous systems.

[18] The best way to understand this is by looking at the "box versions" of these two axioms: $[a]\Box\phi \to \Box[a]\phi$ and $\Box[a]\phi \to [a]\Box\phi$.

However, note that the structure of NM differs from the above NL axiom in the form of the first modality in the consequent: in NM we have a box modality and in NL we have a diamond modality. This is because not all the announcements are executable at the current world. Therefore we do not have $[\psi]\Box\phi \to \Box[\psi]\phi$ which amounts to the following *no learning* structural property:

$$
\begin{tikzcd}
s \arrow[r] & s' \arrow[d, "\psi"] & & s \arrow[r] \arrow[d, "\psi"] & s' \arrow[d, "\psi"] \\
& t' & \overset{\texttt{NL}}{\Longrightarrow} & t \arrow[r] & t'
\end{tikzcd}
$$

Requiring both the above property and propositional invariance in the following extended model (consider the solid relations) would prevent the agent from learning p after the announcement of p (consider truth value of $[p]Kp$ at the upper-right world under the addition of the dashed relations):

$$
\begin{tikzcd}
\neg p \arrow[loop left] \arrow[r, leftrightarrow] \arrow[d, dashed, "p"] & p \arrow[loop right] \arrow[d, "p"] \\
\neg p \arrow[loop left, dashed] \arrow[r, leftrightarrow, dashed] & p \arrow[loop right]
\end{tikzcd}
$$

In fact, NM can be viewed as an axiom of *conditional* no learning: it specifies in what cases the agent *cannot* learn, based on the executability of the announcements. This makes it possible for an agent to learn something via its observation of the announcements. The subtle difference between NM and NL may turn out to be crucial in distinguishing dynamic epistemic logics from the usual epistemic temporal logics discussed before.[19]

Based on the above understanding of the axioms, **PAN** naturally relates the PAL framework to the ETL framework whose models are similar to our extended models with both epistemic and action relations. Van Benthem et. al. (2009) characterize the DEL-generated ETL models (under uniform protocols) by the following properties: *synchronicity, (synchronous) perfect recall, (synchronous) uniform no miracles*, *propositional invariance*, and *bisimulaiton invariance*, where the uniform no miracles property is a more general version of the no miracles property that we have discussed.[20] Dégremont et al. (2011) argue that *synchronicity* is not an intrinsic feature of DEL, but it is introduced by the particular translation used in van Benthem et al. (2009). Our axiomatization **PAN** suggests that perfect recall, uniform no miracles, and propositional invariance are indeed intrinsic to PAL (and to DEL in general).

[19] The first author conjectures that this difference may require new techniques in axiomatizing ETL (with fixed point operators) on structures with properties like no miracles.

[20] Based on this result, Proposition 3 in van Benthem et al. (2009) also gives a characterization of PAL-generated ETL models where the uniform no miracles can be reduced to no miracles.

Moreover, the partial-functionality is another property of PAL which is replaced by a purely *structural property* of bisimulation invariance in van Benthem et al. (2009). To make the above discussion completely precise, we need to axiomatize the event model DEL in the same fashion as **PAN**, which we leave for a future occasion.

To end this section, we would like to mention that there are other approaches of modelling logical dynamics in epistemic temporal frameworks making use of similar axioms as those in **PAN**. For example, Herzig and Lima (2006) developed an epistemic temporal framework for observations and ontic actions which validates many axioms in **PAN**. A follow-up of this work is Aucher and Herzig (2011), where the converse operators are used in an epistemic PDL setting to model even more general logical dynamics than the product updates in Baltag and Moss (2004).

6 Conclusions and future work

We have shown that **PA** and many natural extensions of **PA** are not complete w.r.t. the standard semantics on the class of all the Kripke frames. The same results hold for **PAK** w.r.t. **S5** models. We also gave an alternative axiomatization **PAN** of PAL which involves axioms that naturally define the features of the announcements and the assumptions about the agents. A proof based on the canonical model shows that it is sound and complete via a detour method using an auxiliary semantics.

Technically we have been doing mainly two things in this paper:

1. giving non-standard (context-dependent/auxiliary) semantics for PAL;
2. giving non-standard axiomatizations by making use of non-standard semantics.

There is much more to be explored in the above two lines of research. First of all, the study of context-dependent semantics should be carried on further in DEL. For example, although we focused on the context-dependent semantics that are different from the standard PAL semantics, it is not hard to design an *equivalent* context-dependent semantics for PAL as follows:[21]

$$
\begin{array}{rcl}
\mathcal{M}, s \Vdash \phi & \Leftrightarrow & \mathcal{M}, s \Vdash_{\top} \phi \\
\mathcal{M}, s \Vdash_{\rho} \top & \Leftrightarrow & \text{always} \\
\mathcal{M}, s \Vdash_{\rho} p & \Leftrightarrow & p \in V(s) \\
\mathcal{M}, s \Vdash_{\rho} \neg\phi & \Leftrightarrow & \mathcal{M}, s \nVdash_{\rho} \phi \\
\mathcal{M}, s \Vdash_{\rho} \phi \wedge \psi & \Leftrightarrow & \mathcal{M}, s \Vdash_{\rho} \phi \text{ and } \mathcal{M}, s \Vdash_{\rho} \psi \\
\mathcal{M}, s \Vdash_{\rho} \Box\psi & \Leftrightarrow & \forall t : (s \to t \text{ and } \mathcal{M}, t \Vdash_{\top} \rho) \text{ implies } \mathcal{M}, t \Vdash_{\rho} \psi \\
\mathcal{M}, s \Vdash_{\rho} [\psi]\phi & \Leftrightarrow & \mathcal{M}, s \Vdash_{\rho} \psi \text{ implies } \mathcal{M}, s \Vdash_{\rho \wedge [\rho]\psi} \phi
\end{array}
$$

Instead of transforming the models, we may well just change the explicit context during the process of evaluation. This context-dependent semantics seems to be closer to the spirit of (Stalnaker 1978) which inspired the study of update semantics and dynamic epistemic logics. As we have demonstrated in the first part of the paper, the context-dependent semantics gives us more flexibility in designing the exact update

[21] Note that this semantics is very similar to the first non-standard semantics given in Sect. 3 except the clause for $[\psi]\phi$. Here we use the context accumulation inspired by the composition axiom.

mechanisms (cf. Remark 3). Moreover, alternative semantics may help us to make use of the existing techniques of modal logic and potentially unify different approaches to logical dynamics in terms of different context updating policies.

For the second line of research, note that the proof strategy (Sect. 4.2) that we used to show the completeness of **PAN** is a very general one. Thus similar analysis should apply to many existing dynamic epistemic logics (e.g., the event model approach: Baltag and Moss 2004). With such proof techniques we may go beyond the "reducible logics", especially when we relax the restriction that the updated epistemic structure should be computed from the current *epistemic* information *only* (cf. Wang and Li 2012 and Hoshi 2009). There is also some hope in getting general completeness results for DEL-style logics: note that the axioms we use here are Sahlqvist formulas, as observed by van Benthem (2012), so according to our proof strategy, the remaining work is to show the equivalence of the auxiliary semantics and standard semantics on the extended models satisfying the corresponding (first-order) properties of the axioms. Finally, it is not clear whether our method will help in axiomatizaing PAL with common knowledge. We leave such analyses for future work.

Based on the technical results and previous discussions, we want to stress the following points in the end:

- Axiomatizating PAL and other DEL logics are more subtle than they may look, which invites careful investigations.
- Studying alternative semantics and axiomatizations pays back in giving us better understandings about the DEL framework and its relevant results. The semantics of DEL logics may differ a lot from each other, but from the point of view of proof systems, the common features may emerge more clearly, if appropriate axioms are used.
- On the other hand, we may also deviate from the commonly used axiom schemata such as `NM` and `PR`, which amount to different assumptions about agents (e.g., agents without perfect recall and no miracles).[22]
- There are different ways to conduct the reductions in DEL logics which require different facilities in the proof system. It is crucial to decide carefully which way you want to take and verify whether it is indeed possible.[23]
- The reduction phenomenon in DEL logics is definitely a blessing with deep mathematical roots and fruitful applications. However, it comes with some theoretical limitations as well, as we discussed in Sect. 5. Since the reduction is not the goal of our research in logical dynamics, we may intentionally relax the conditions that are necessary for the reductions and, as already proposed in van Benthem (2011, Chap. 11), consider models richer than the purely epistemic ones. We hope this may lead us to a broader view of logical dynamics.

[22] For example, Liu (2008) discussed memory-less agents.

[23] We have shown two general ways to reduce PAL to EL: "inside-out" (by using `RE`) and "outside-in" (by using `!COMP`). The composition axiom plays an important role when `RE` is not available as we have shown in Theorem 24. In some other cases, the composition may not be possible but `RE` is available cf. e.g., van Benthem and Minică (2009).

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